Université de Montréal

Essais en microéconomie théorique et appliquée

par
Éric Bahel

Département de sciences économiques
Faculté des arts et des sciences

Thèse présentée à la Faculté des études supérieures
en vue de l'obtention du grade de Philosophiæ Doctor (Ph.D.)
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Essais en microéconomie théorique et appliquée

présentée par:

Éric Bahel

a été évaluée par un jury composé des personnes suivantes:

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RÉSUMÉ

Cette thèse comporte trois essais en microéconomie théorique et appliquée aux ressources naturelles. Le premier essai s'intéresse aux problèmes de partage des coûts, et plus particulièrement à l'étude de l'axiome "Ranking". Les deux autres essais sont des applications de la théorie économique à des problèmes rencontrés sur le marché de l'énergie.

Dans le premier chapitre, "The Implications of the Ranking Axiom for Discrete Cost Sharing Methods", nous étudions l'axiome "Ranking" rencontré dans les problèmes de partage de coûts. Considérons un groupe d'agents qui partagent le coût d'un projet commun, en fonction des demandes respectives formulées par ces agents vis-à-vis dudit projet. L'axiome "Ranking" requiert que les parts de coût respectent le même ordre que les demandes formulées par les agents, pour toute fonction de coût symétrique. En d'autres termes, pour des agents similaires, celui qui a la demande la plus élevée devrait payer une plus grande part du coût. Ce chapitre vise à déterminer l'ensemble des méthodes discrètes satisfaisant "Ranking". Pour que cette propriété soit satisfaite, nous montrons que le flot définissant la méthode doit vérifier une propriété particulière de symétrie. Dans le cas du partage avec trois agents et plus, nous établissons des conditions nécessaires de symétrie ; toutefois, nous montrons qu'elles ne sont plus suffisantes. Nous déterminons aussi toutes les méthodes basées sur des flots fixes élémentaires qui vérifient notre propriété.

Le second chapitre, "Optimal Management of Strategic Reserves of Nonrenewable Natural Resources", examine la question de la gestion des réserves stratégiques de pétrole. En effet, afin de se prémunir contre les ruptures d'approvisionnement (embargos, guerres, etc.), de nombreux pays détiennent des réserves de pétrole. Faisant l'hypothèse que le prix du pétrole évolue suivant la règle de Hotelling, nous déterminons la politique optimale de gestion des stocks stratégiques de pétrole pour un pays qui fait face à la menace d'un embargo dont la date d'occurrence et la durée sont aléatoires. Nous montrons l'exis-

tence d'une trajectoire décroissante (pour les réserves) que le pays veut atteindre afin de se protéger contre les ruptures d'approvisionnement. Le pays importateur a également la possibilité d'investir dans la recherche d'une autre source d'énergie (renouvelable), ceci afin de se libérer de la menace d'embargo. Le succès de cette recherche est aléatoire, ainsi que la date à laquelle le pays aura à sa disposition la nouvelle source d'énergie. Nous montrons l'existence d'un niveau d'investissement optimal dans la recherche. Ce dernier décroît avec la taille des réserves stratégiques.

Pour ce qui est du troisième chapitre, "The Economics of Oil, Biofuel and Food Commodities", nous nous intéressons à la modélisation des effets de l'introduction des biocarburants sur l'offre de produits agricoles de base. Nous considérons une économie constituée d'un cartel pétrolier et d'agriculteurs qui, en plus de fournir des biens alimentaires, produisent aussi de l'énergie sous forme de biocarburant. Les terres agricoles sont partagées entre ces deux activités de production. Dans un premier temps, nous dérivons de façon explicite la relation entre le prix de l'énergie et celui des aliments. Ensuite, nous déterminons le sentier optimal d'extraction du cartel, ainsi que les sentiers de prix. Nous montrons que le prix des produits alimentaires croît pendant la phase d'extraction, indépendamment de ce que la population est croissante ou constante. Dans le cas d'une population croissante, le prix des aliments continue d'augmenter après l'épuisement du pétrole.

Mots-clés : méthode de partage des coûts, "Ranking", flot, profil de demande, symétrie, importations, resource non renouvelable, embargos aléatoires, réserves stratégiques, biocarburants, épuisement du pétrole, prix de l'énergie, prix des aliments.

ABSTRACT

This thesis consists of three essays. The first one deals with the theory of cost sharing. We study the implications of "Ranking", which is a fairness requirement. The second and third essays are applications of microeconomic theory to the analysis of issues encountered in the energy market.

In the first chapter, "The Implications of the Ranking Axiom for Discrete Cost Sharing Methods", we study the Ranking axiom in cost sharing problems. Suppose that a group of agents have to share the cost of a joint project, depending on their respective demands. The Ranking axiom requires the demands and the cost shares to have the same ordering, as long as the cost function is symmetric in these demands. In other words, if the demands of two agents play exactly the same role in raising the cost of the project, then the agent who demands more should pay more. This is a legitimate fairness condition. Dealing with the discrete version of the model, we characterize the set of all methods satisfying Ranking in the case of two agents. We prove that, in order to satisfy our requirement, the flow representing the method must exhibit a specific symmetry condition. With three agents or more, we derive strong necessary conditions. We show, however, that they are not sufficient. We also characterize the elementary fixed-flow methods satisfying the axiom.

The second chapter, "Optimal Management of Strategic Reserves of Nonrenewable Natural Resources", examines the issue of strategic petroleum reserves management. Indeed, due to uncertainty on the supply side, many countries are holding precautionary stocks of oil. Assuming that the evolution of the price of oil is consistent with the Hotelling rule, we derive the optimal stockpiling policy for a small country which is likely to suffer an embargo, the occurrence and duration of which are stochastic. We show the existence of a decreasing "target path" (for the reserves) that the country wants to attain in order to hedge against these disruptions. We also introduce the possibility for the importing country to invest in research on a backstop technology, which would reduce

the dependence on oil. Success in research is random and the date at which the backstop will be discovered is also stochastic. We prove that it is always optimal for the importing country to undertake research in order to free itself from the embargo threat and achieve energy independence. The effort invested in research is shown to decrease with the size of the strategic reserves.

As for the third chapter, 'The Economics of Oil, Biofuel and Food Commodities", it studies within a tractable model the effects on the food market of the introduction of biofuels as a substitute for fossil fuel in the energy market. We consider a world economy with an oil cartel and a competitive fringe of farmers producing energy in the form of biofuels. Farmers also produce food and sell it on the world food market. We determine the resulting relationship between prices in the energy and food markets and characterize the cartel's extraction path and the price path of energy. We then show that the price of food will be growing as long the oil stock is being depleted, whether population is growing or not, and that it will keep growing after the oil stock is exhausted if population is growing.

Keywords : cost sharing method, Ranking, demand profile, flow, symmetry, imports, nonrenewable resource, random embargoes, strategic reserves, biofuel, oil depletion, energy price, food price.

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À mes chers parents, Madeleine et Zachée Bahel

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INTRODUCTION GÉNÉRALE

Cette thèse est constituée de trois essais en microéconomie. Le premier est un essai théorique qui s'intéresse aux problèmes de partage des coûts. Il étudie les implications de l'axiome "Ranking", qui constitue une exigence d'équité dans le partage. Les deuxième et troisième essais sont des applications de la théorie microéconomique aux ressources naturelles. Ils examinent des questions rencontrées sur le marché de l'énergie, et plus spécifiquement sur le marché du pétrole.

Dans le premier chapitre, nous considérons le modèle classique de partage des coûts qui vise à répartir le coût total d'un projet commun entre les différents agents impliqués. Les agents formulent leurs demandes qui sont des quantités positives de biens possiblement hétérogènes, le coût du projet étant une fonction croissante de ces demandes. Une méthode de partage des coûts est une règle qui, étant donné les demandes des différents agents et la fonction de coût du projet, détermine les parts de coût respectives des agents de façon à satisfaire l'équilibre budgétaire : la somme des parts de coût des agents est égale au coût total du projet. A noter qu'il existe deux versions du modèle de partage des coûts. La version continue, dans laquelle les demandes des agents sont représentées par des nombres réels, a été introduite par Aumann et Shapley (1974). Quant à la version discrète, qui a été présentée pour la première fois par Moulin (1995), elle est caractérisée par le fait que les demandes des agents sont représentées par des entiers naturels.

De nombreux travaux ont contribué à développer une abondante littérature liée aux problèmes de partage des coûts. Sur le plan de l'axiomatisation, afin que les règles de partage proposées soient stables et équitables, plusieurs propriétés ont été introduites et étudiées. Par exemple, pour un projet constitué de plusieurs phases (production, stockage, livraison, administration,...), l'axiome d'Additivité requiert que les parts de coûts du projet global soient égales à la somme des parts de coûts spécifiques à chacune des parties du projet. Aussi, l'axiome "Dummy" stipule qu'un agent dont la demande n'affecte en aucune façon le coût du projet devrait se voir attribuer une part de coût nulle.

Au nombre des règles de partage les plus utilisées, mentionnons la méthode de Shapley-Shubik (1962), la méthode d'Aumann-Shapley (1974) et la méthode sérielle de Moulin et Shenker (1992). Le lecteur intéressé peut par exemple consulter Moulin (2002) pour une revue détaillée des axiomes et méthodes de partage.

A côté de cette recherche axiomatique, des applications de plus en plus nombreuses de la théorie du partage des coûts ont été développées : Billera, Heath et Raanan (1978) utilisent la théorie des jeux non atomiques pour proposer une tarification équitable des différents types de communication téléphonique. D'autres applications intéressantes ont été développées, notamment pour les problèmes : d'exploitation commune d'une ressource (Case [1979], Sharkey [1982]), de production de biens publics (Moulin [1994]), d'accès à un réseau (Feigenbaum *et al* [2004]), de congestion et files d'attente (Chen et Zhang [2005]).

Le premier chapitre de cette thèse s'inscrit dans le courant axiomatique de cette littérature. Il étudie de façon spécifique la propriété "Ranking" pour les mécanismes de partage des coûts. Lorsque des agents affectent le coût d'un projet de façon identique (c'est-à-dire si la fonction de coût est symétrique par rapport aux demandes de ces agents), cette propriété permet de comparer les parts de coût attribuées aux agents. Plus précisément, si la fonction de coût est symétrique par rapport aux demandes de deux agents donnés, alors celui de ces agents qui a la demande la plus élevée doit payer une plus grande part du coût total. Cette propriété traduit donc une exigence d'équité et de justice dans le partage. Il est à noter qu'elle implique l'exigence de symétrie introduite par Shapley (1953), qui veut que des agents symétriques ayant le même niveau de demande paient des parts identiques du coût total.

L'axiome "Ranking", qui est présenté et commenté par Moulin et Sprumont (2005), n'avait jusqu'ici pas été étudié de façon spécifique. Le premier chapitre de la thèse, utilisant la version discrète du modèle, a pour objectif de caractériser l'ensemble des méthodes de partage qui satisfont cette exigence d'équité. Comme il est de coutume dans cette littérature, nous considérons des méthodes de partage qui vérifient les axiomes Additivité et "Dummy". Grâce au résultat de Moulin et Vohra (2003), il est établi que l'ensemble des mécanismes de partage vérifiant ces deux propriétés correspond à la famille des méthodes de flot. Ces méthodes de flot sont des combinaisons convexes de méthodes dites "path generated" qui sont elles-mêmes des règles de partage qui déterminent la part de coût d'un agent en additionnant, le long d'un chemin prédéterminé, les incréments de la fonction de coût dus à cet agent (voir par exemple Sprumont [2008]).

Pour ce qui est du modèle de partage de coûts avec deux agents, nous montrons dans la Section 2 de ce premier chapitre que l'axiome "Ranking" est caractérisé par la symétrie du flot. Plus précisément, une méthode de flot satisfait "Ranking" si et seulement si le flot qui la représente est symétrique à l'intérieur du carré défini par la plus petite des deux demandes. Le comportement du flot à l'extérieur de ce carré n'influence pas le fait que la méthode satisfait ou non la propriété. Dans le modèle général où le nombre d'agents est quelconque, nous établissons une condition nécessaire de symétrie qui généralise la propriété mise en évidence dans le cas de deux agents. Toutefois, nous montrons qu'avec trois agents ou plus, cette propriété de symétrie n'est plus suffisante pour garantir que la méthode de partage vérifie "Ranking". Finalement, dans la Section 4 de ce premier essai, nous examinons le cas des flots fixes élémentaires. Un flot appartient à cette sous-famille s'il peut être exprimé comme la moyenne de toutes les méthodes de chemin ("path generated") générées par les permutations d'un chemin donné. Pour le cas spécifique de ces flots fixes élémentaires, un résultat de caractérisation est proposé.

Les deux autres essais qui composent la thèse utilisent des outils de l'économie des ressources naturelles. Étant donné qu'ils examinent tous les deux des problèmes dynamiques, ils font notamment appel à la théorie du contrôle optimal et au principe du maximum de Pontryagin.

Le deuxième chapitre de la thèse s'intéresse à la gestion des réserves pour un pays qui importe une ressource naturelle non renouvelable et qui fait face à la menace permanente d'une rupture d'approvisionnement. L'exemple typique est celui d'un pays qui importe du pétrole, l'acquisition de cette ressource étant soumise à des aléas (embargos, guerres, ouragans, etc.) qui, lorsqu'ils surviennent, entraînent la cessation des importations pour une période indéterminée. Pour se prémunir contre ces interruptions aléatoires, de plus en plus de pays détiennent des réserves stratégiques de pétrole. L'Agence Internationale de l'Énergie (AIE) recommande même à ses pays membres de maintenir des stocks de précaution équivalents à au moins 90 jours de consommation de pétrole. Il est à noter qu'avec l'amélioration des méthodes de condensation et de transport, plusieurs pays ont également entrepris de construire des réserves stratégiques de gaz naturel, qui est stocké notamment sous forme liquide.

Quelques travaux de la littérature se sont penchés sur la question de la gestion des réserves pour un bien reproductible (voir par exemple Loury [1983]). Dans le deuxième chapitre de la thèse, nous traitons du cas spécifique d'une ressource non renouvelable. Cette particularité a des implications importantes, notamment en ce qui concerne l'évolution du prix de la ressource. À la suite des deux embargos pétroliers des années 70, plusieurs auteurs ont proposé des modèles visant à déterminer la politique optimale de gestion des réserves stratégiques de pétrole. Teisberg (1981) présente un modèle de programmation dynamique dans le but de déterminer la stratégie optimale d'acquisition et de stockage de pétrole pour les États-Unis. Pour un pays qui est en même temps importateur et producteur de pétrole, Hillman et Long (1983) déterminent la tarification et l'utilisation optimale des ressources locales pendant un embargo qui dure indéfiniment. Ils montrent que pendant l'embargo, l'industrie locale doit, de façon optimale, conserver la parité avec le cours mondial du pétrole. Finalement, Bergström, Loury et Persson (1985) étudient la question de la gestion optimale des stocks de pétrole pour un pays qui fait face à la possibilité d'embargos répétitifs, entrecoupés de périodes d'échange.

Le modèle utilisé est proche de celui de Bergström, Loury et Persson à l'exception notamment du fait que, contrairement à eux, nous ne supposons pas que le prix de la ressource est constant dans le temps. Nous adoptons l'hypothèse selon laquelle le prix de la ressource évolue suivant la règle de Hotelling pour les ressources non renouvelables (Hotelling [1931]). Compte tenu de cette hypothèse, nous déterminons la stratégie optimale de constitution des réserves stratégiques pour le pays importateur et aussi l'utilisation qui doit en être faite durant un embargo. Nous montrons clairement qu'il existe une phase de constitution des réserves pendant laquelle le pays accroît son stock. Une fois cette phase terminée, le pays à intérêt à consommer une partie de son stock, même s'il n'est pas sous embargo. A l'aide d'une formule de récurrence, nous proposons une approche pour déterminer la stratégie optimale de gestion des réserves pour le cas de plusieurs embargos séparés par des périodes d'importation.

Finalement, comme dans l'article de Dasgupta et Heal (1974), le pays importateur de pétrole a la possibilité d'investir dans la recherche d'une source d'énergie renouvelable. La découverte de cette source d'énergie survient à une date aléatoire ; à partir de cette date, le pays cesse ses importations de pétrole et n'est donc plus sujet à la menace d'embargo. Les résultats obtenus dans la Section 4 de ce chapitre montrent que le pays importateur a toujours intérêt à investir dans la recherche d'un substitut aux importations de pétroles. L'incitation à investir dans cette recherche est d'autant plus forte que le niveau des réserves stratégiques est faible.

Le troisième chapitre de la thèse relie la récente flambée des prix des produits alimentaires de base à l'introduction et au développement des biocarburants. Au courant de la période 1999-2008, les prix alimentaires ont connu un accroissement sans précédent au point de déclencher des mouvements de contestation et même des émeutes dans de nombreux pays. Le pic d'inflation a été atteint au courant de la période 2007-2008 qui, selon la FAO ("Food and Agriculture Organization"), a enregistré une montée des prix des denrées alimentaires de 52%. Ceci a conduit à la tenue d'un sommet mondial de l'alimentation en juin 2008 à Rome, sous l'égide des Nations Unies.

Plusieurs théories ont été proposées pour expliquer cette envolée des prix alimentaires. La plus plausible, qui est celle que mettons en évidence dans le troisième essai de cette thèse, est que le marché des produits agricoles est de plus en plus influencé par l'évolution du cours du pétrole. En effet, au courant de la dernière décennie, le cours élevé du pétrole a alimenté la forte expansion du secteur des biocarburants. La conséquence en est que de plus en plus de ressources initialement consacrées à la production des denrées alimentaires (riz, blé, orge, maïs, huile, etc.) sont maintenant affectées à la production de biocarburants. Selon l'IFPRI ("International Food Policy Research Institute"), l'introduction et le développement des biocarburants expliquerait près du tiers de l'envolée des prix des produits alimentaires de base. L'objectif du troisième essai est la mise au point d'un cadre d'analyse dynamique intégrant à la fois le marché de l'énergie (pétrole, biocarburant) et celui des denrées alimentaires et permettant de faire des prédictions sur l'évolution des prix dans ces deux marchés.

Compte tenu du caractère récent de la question étudiée, la littérature qui s'y rattache n'est pas très fournie. Dans une revue détaillée, Rajagopal et Zilberman (2007) présentent les différentes implications environnementales, économiques et politiques des biocarburants. Ces deux auteurs soulignent le fait que l'activité de production liée à la génération actuelle de biocarburants utilise de façon intensive des ressources telle que la terre, l'énergie et les intrants chimiques. Ils mentionnent également le fait que la littérature est concentrée sur les discussions reliées à la réduction des émissions de CO_2, alors que des questions telles que l'effet du développement des biocarburants sur la santé, la biodiversité et les ressources (terre, eau, etc.) méritent également d'être examinées. Hochman, Sexton et Zilberman (2008), dans un modèle d'équilibre partiel, montrent que la régulation du secteur des biocarburants peut être utilisée dans le but d'améliorer le bien-être des agriculteurs. Ils discutent également des effets de l'innovation (dans le secteur des biocarburants) sur les marchés respectifs de l'énergie et des produits alimentaires.

Dans le troisième chapitre de la thèse, nous considérons des agriculteurs qui, en plus de produire des biens alimentaires, se comportent comme une frange concurrentielle vis-à-vis du cartel pétrolier. Le cartel fixe le prix de l'énergie et les agriculteurs déterminent ensuite leurs offres sur les deux marchés. L'analyse intègre formellement l'effet de la croissance de la population sur les demandes dans les deux marchés. La Section 2 du troisième chapitre établit explicitement la corrélation positive entre le prix de l'énergie et celui des biens alimentaires. Dans la Section 3, nous formulons le problème dynamique du cartel et dérivons son sentier optimal d'extraction. Le pétrole est épuisé en temps fini et le sentier d'extraction, même s'il peut être croissant dans un premier temps à cause de l'accroissement de la population, finit par décroître pour atteindre zéro à la date terminale d'extraction. Finalement, les sentiers de prix sont explicitement déterminés : nous montrons que l'épuisement du pétrole et la croissance de la population sont les facteurs explicatifs de la hausse continue du prix de l'énergie et, par ricochet, de celui des aliments.

CHAPITRE 1

THE IMPLICATIONS OF THE RANKING AXIOM FOR DISCRETE COST SHARING METHODS

1.1 Introduction

We consider the cost sharing model where the cost of a project is to be be split between the different agents that are involved, depending on their respective demands. Agents' demands are quantities of possibly distinct goods. In this literature, the seminal paper is by Shapley (1953) and is based on cooperative games, the simplest models in the cost sharing theory. For these cooperative games, the different agents have to state whether or not they want to participate in the project. Hence, their demands take only two values : 0 or 1. For such problems, the *Shapley value* provides a very consistent solution. Other methods have been proposed in the general model. Using the notion of the stand-alone cost, Shubik (1962), adapted the Shapley value to the case where demands are real numbers. Aumann and Shapley (1974) extended the Shapley value to non-atomic games. Also, Moulin and Shenker (1992) introduced *serial cost sharing*.

Along with these cost sharing methods (CSM), several axioms have been studied as requirements that a mechanism should satisfy in order to be fair and consistent. Requiring the allocation of the cost shares to satisfy some desirable properties narrows the set of admissible sharing methods. For instance, on the subset of binary-demand problems, the Shapley method is the only one satisfying additivity, dummy and symmetry (Shapley [1953]).

Dealing with the discrete version of the model, we examine the implications of the Ranking property. We would like to compare the shares of agents who affect the cost of the project in the same way (i.e. the cost function is symmetric in these agents' demands). Suppose that the cost of the project is symmetric with respect to the demands of two given agents. The Ranking axiom says that the agent who demands more should not be charged a lower share. Among a group of agents who affect the cost of the project in the same manner, whoever demands more should pay more. Hence, studying this property is desirable if we want the sharing mechanism to be fair. In the $0-1$ model, with nonnegative cost shares, Ranking is trivially satisfied by all symmetric methods satisfying the property that all agents demanding 0 pay nothing, which is a consequence of additivity and dummy.

However, in the general model where demands are integer-valued, some widely used methods (the Aumann-Shapley pricing for instance) fail to meet Ranking.[1] Indeed, discrete Aumann-Shapley pricing requires that the shares be computed using the Shapley value of the game where each unit of each good is considered as a single player (see Sprumont (2005)). To see why this method violates Ranking, consider the simple example where the cost of serving the demands z_1 and z_2 is given by $C(z_1,z_2) = \min\{z_1,z_2\}$ and the demand profile is $q = (2,1)$. Computing the Shapley value of the corresponding (three players') game yields $y_1 = 1/3$ and $y_1 = 2/3$. Thus, although agent 1 demands more, she pays less.

We consider sharing mechanisms satisfying the two basic requirements additivity and dummy, which are very well admitted in this literature. From Moulin and Vohra (2003), we know that a discrete CSM meeting these two properties can be represented by a flow system.

In the problem with two agents, we show that Ranking is equivalent to a specific symmetry property induced on the flow. In higher dimensions (with three agents or more), we also identify a property of the flow which generalizes the result with two agents. However, we show that it is not sufficient for a characterization result. We propose a

[1]For a detailed discussion on the Ranking property, see Moulin and Sprumont (2005). They consider both a weak and a strong version of this axiom, the latter defining the *partial responsibility theory*, where agents are not held responsible for the asymmetries of the cost function. In the present paper, we study the weak version of Ranking which is related to the *full responsibility theory* : agents are accountable for the idiosyncrasies of the cost function.

characterization result for the so-called elementary fixed flows.

The paper unfolds as follows. Section 2 presents the model ; we formally introduce the Ranking axiom and provide some (counter) examples. Section 3 presents the results ; we discuss two cases : the model with two agents and the model with three agents or more. Section 4 examines the specific case of elementary fixed flows. Finally, Section 5 concludes with some comments.

1.2 The Model

1.2.1 Definition of a CSM

$N = \{1,..,n\}$ is the set of agents (with $n \in \mathbb{N}$). For $x,y \in \mathbb{N}^n$, we denote by $[x,y]$ the set $\{z \in \mathbb{N}^n \text{ s.t. } x_i \le z_i \le y_i\}$. The integer-valued demands of the agents are represented by the profile $q \in [0_n, (\bar{q},..,\bar{q})]$, where $\bar{q}$ might be infinite. The cost of serving these demands is $C(q)$ where $C : [0_n, (\bar{q},..,\bar{q})] \rightarrow \mathbb{R}$ is a *nondecreasing* function satisfying $C(0_n) = 0$. Let $\mathscr{C}(N)$ be the set of all such functions. Define σ_{ij}, the transposition relative to $i,j \in N$, as $\sigma_{ij}(z) = (\underbrace{z_j}_{i}, \underbrace{z_i}_{j}, z_{-ij})$, for any $z \in \mathbb{N}^n$. The cost function $C \in \mathscr{C}(N)$ is said to be $i-j$ symmetric if $C(z) = C(\sigma_{ij}(z))$, for all $z \in [0_n, (\bar{q},..,\bar{q})]$.

Definition 1.2.1. Given N, a *Cost Sharing Method* (CSM) φ is a mapping defined from $[0_n, (\bar{q},..,\bar{q})] \times \mathscr{C}(N)$ to $\mathbb{R}^n_+$ such that :

$$\forall\, C \in \mathscr{C}(N), \forall\, q \in [0_n, (\bar{q},..,\bar{q})], \ \sum_{i \in N} \varphi_i(q,C) = C(q)\ . \tag{1.1}$$

A CSM is thus a mechanism allowing to compute the different shares of the agents, according to their demands and the cost function. Notice that we assume nonnegative cost shares.[2]

[2] In some cases, like in the provision of public goods, it might make sense to allow for negative cost shares. See for instance De Frutos (1998) and Moulin (2002).

1.2.2 The axioms

We essentially deal with three axioms.[3]

- Axiom 1 : A CSM φ meets *additivity* if

  $\forall\ C_1, C_2 \in \mathscr{C}(N)$, $\forall\ q \in [0_n, (\bar{q}, .., \bar{q})]$, we have : $\varphi(q, C_1 + C_2) = \varphi(q, C_1) + \varphi(q, C_2)$.

Additivity says that if the project is constituted by several components, then the overall share for each agent has to be equal to the sum of component-specific shares. This axiom might be very convenient in practice.

For $i \in N$, let e_i be the the vector of which all coordinates are 0, except for the i-th one which is 1 (i.e. $e_{ii} = 1$ and $e_{ij} = 0$, for $j \in N \setminus i$). Denote by ∂C_i the function defined as : $\partial C_i(z) = C(z) - C(z - e_i)$, for any $z \in [0_n, (\bar{q}, .., \bar{q})]$ such that $z_i \geq 1$.

- Axiom 2 : A CSM φ meets *dummy* if

  $\forall\ C \in \mathscr{C}(N), \forall\ i \in N$, we have : $[\partial_i C = 0] \Longrightarrow [\varphi_i(q, C) = 0, \forall\ q \in [0_n, (\bar{q}, .., \bar{q})]$.

Dummy requires that an agent be charged zero cost if she is not responsible for the increases in the cost (i.e the cost function does not depend on this agent's demand).

The two preceding axioms are almost unanimously admitted in the literature. The characterization result by Moulin and Vohra (2003) describes the set of methods satisfying both additivity and dummy.

Definition 1.2.2. $\diamond$ A *flow to a demand profile* $q \in [0_n, (\bar{q}, .., \bar{q})]$ is a mapping $f^q : [0, q] \rightarrow \mathbb{R}^n_+$ such that :

$$f_i^q(z) = 0 \quad if \quad z_i = 0 \tag{1.2}$$

$$\sum_{i \in N(0,q)} f_i^q(e_i) = 1 \tag{1.3}$$

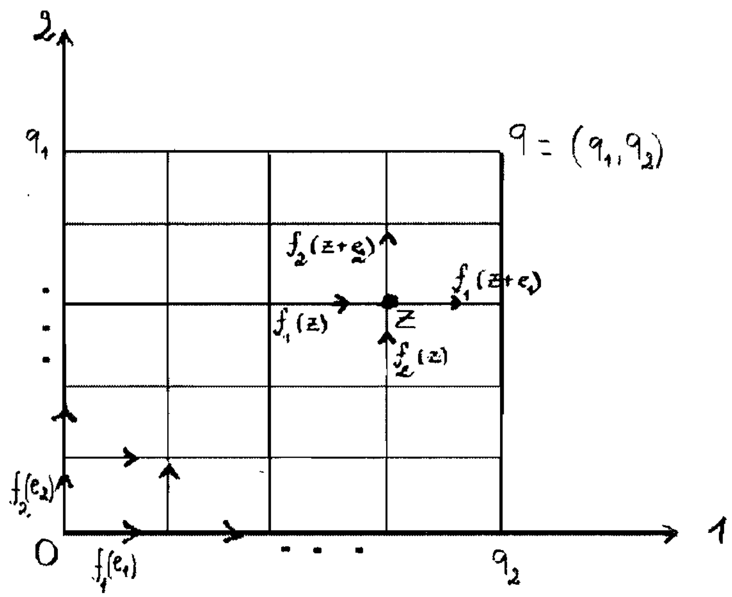
$$\sum_{i \in N(0,q)} f_i^q(z) = \sum_{i \in N(z,q)} f_i^q(z + e_i) \ \ \forall\ z \in]0, q], \tag{1.4}$$

where $N(z,q) = \{i \in N |\ z_i < q_i\}$.

[3]For a detailed review of the axioms in this field, see for example Moulin and Sprumont (2007).

◇ A *flow (system)* is a list $f = \{f^q(.),\ q \in [0_n, (\bar{q}, .., \bar{q})]$ where $f^q(.)$ is a flow to q.

FIG. *1.1 – Illustration of a flow*



Flow conservation : $f_1^q(z) + f_2^q(z) = f_1^q(z+e_1) + f_2^q(z+e_2)$.

Equation (1.4) represents the so-called flow conservation property guaranteeing, at each node z, that the sum of all incoming flows be equal to the sum of all exiting flows (see Figure 1.1). In general, the flows to two different profiles q and q' are not necessarily related. However, for the class of *fixed flows*, the flows to two comparable profiles relate. This family of flows was first introduced in Moulin and Sprumont (2004).

Definition 1.2.3. *Flow methods*

A CSM φ is said to be a *flow method* if it is represented by some flow f, that is :

$$\forall\, C \in \mathscr{C}(N), \forall\, q \in [0_n, (\bar{q}, .., \bar{q})] : \ \varphi_i(q, C) = \sum_{z \in [e_i, q]} f_i^q(z)\ \partial_i C(z)\,. \tag{1.5}$$

Note that for any cost function C and any demand profile q, the share of agent i obtains by computing the scalar product of the ith component of the flow to q, f_i^q, and the vector of cost increments, $\partial_i C$.

Lemma 1.2.1. Moulin and Vohra (2003)

A CSM satisfies additivity and dummy if and only if it is a flow method.

This result points out that the class of flow methods is precisely the set of all discrete CSMs satisfying additivity and dummy. As it is generally the case in the literature, we assume these two axioms. Hence, we focus on flow methods.

– *Axiom 3* : A CSM φ meets *Ranking* if : $\forall\, q \in [0_n, (\bar{q}, .., \bar{q})]$ and $\forall\, C \in \mathscr{C}(N)$ that is $i-j$ symmetric, we have : $(q_i \leq q_j) \Rightarrow (\varphi_i(q,C) \leq \varphi_j(q,C))$.

In words, within any subset of identical agents, whoever demands more should pay more. The allocation of the cost shares would be inequitable to some agents if this condition was violated. Hence, the Ranking axiom is a fairness condition that the sharing mechanism should satisfy.

Notice that for a cost function C which is symmetric with respect to i and j, Ranking requires that : $(q_i = q_j) \Rightarrow (\varphi_i(q,C) = \varphi_j(q,C))$ (which is often referred to as *Equal Treatment of Equals*). Two symmetric agents demanding the same amount should be charged exactly the same price.

Although Ranking seems to be a very natural condition, it is violated by some widely used methods.

Example 1.2.1. *Ranking and the Aumann-Shapley method*

As already mentioned, Aumann-Shapley pricing violates Ranking. Consider the cost function $C(z_1,z_2) = min\{z_1,z_2\}$ and the demand profile $q = (q_1,q_2)$ with $q_1 < q_2$. To compute the cost shares, we have to consider the game where each of the units demanded by each one of the two agents is regarded as a single player. Hence, we have a cooperative game with $q_1 + q_2$ players and the *Shapley value* requires that each unit i demanded by agent 1 be charged the cost

$$\varphi_{1i}(q,C)=\frac{1}{q_1+q_2}\sum_{z_1=0}^{q_1-1}\sum_{z_2=0}^{q_2}\frac{\binom{q_1-1}{z_1}\binom{q_2}{z_2}}{\binom{q_1+q_2-1}{z_1+z_2}}[C(z_1+1,z_2)-C(z_1,z_2)]\ \forall\, i=1,..,q_1,$$

where $\binom{m}{p}$ stands for the number of p-combinations of a set of m elements. Since $C(z_1,z_2)=min\{z_1,z_2\}$, only profiles such that $z_1<z_2$ will have their cost increased (by one) following an additional unit demanded by agent 1. Indeed, if $z_1 \geq z_2$, the marginal cost due to agent 1 is null. Hence,

$$\begin{aligned}\varphi_{1i}(q,C) &= \frac{1}{q_1+q_2}\sum_{s=1}^{q_1+q_2-1}\sum_{\substack{k<q_1\\ k<s/2}}\frac{\binom{q_1-1}{k}\binom{q_2}{s-k}}{\binom{q_1+q_2-1}{s}} \qquad \forall\, i=1,..,q_1\\ &= \frac{1}{q_1+q_2}\sum_{s=1}^{q_1+q_2-1}\sum_{\substack{k<q_1\\ k<s/2}}\underbrace{\frac{s!q_2!(q_1-1)!(q_1+q_2-s-1)!}{k!(s-k)!(q_1+q_2-1)!(q_1-1-k)!(q_2-s+k)!}}_{A_{sk}}.\end{aligned}$$

Notice that in this expression, for any coalition of s units, agent 1 has less units than agent 2 ($k<s-k$ since $k<s/2$). Finally, the cost share paid by agent 1 is :
$\varphi_1(q,C)=\sum_{i=1}^{q_1}\varphi_{1i}(C)=\frac{q_1}{q_1+q_2}\sum_{s=0}^{q_1+q_2-1}\sum_{\substack{k<q_1\\ k<s/2}}A_{sk}.$

Similarly, it can be shown that the cost share of agent 2 is given by :

$$\varphi_2(q,C)=\frac{q_2}{q_1+q_2}\sum_{s=1}^{q_1+q_2-1}\sum_{\substack{k<s/2\\ s-k\leq q_1}}\underbrace{\frac{s!q_1!(q_2-1)!(q_1+q_2-s-1)!}{k!(s-k)!(q_1+q_2-1)!(q_2-1-k)!(q_1-s+k)!}}_{B_{sk}}.$$

For $s = 1,..,q_1+q_2-1$ and $s-q_1 \leq k < s/2$, one can easily check that $\frac{A_{sk}}{B_{sk}} = \frac{q_2}{q_1} \prod_{i=k}^{s-k-1} \frac{q_2-1-i}{q_1-1-i} > \frac{q_2}{q_1}$ (recall that $k < s-k$ and $q_1 < q_2$). It follows then that :

$$\varphi_1(q,C) = \frac{q_1}{q_1+q_2} \sum_{s=1}^{q_1+q_2-1} \sum_{\substack{k<s/2 \\ k<q_1}} A_{sk} > \frac{q_1}{q_1+q_2} \sum_{s=1}^{q_1+q_2-1} \sum_{\substack{k<s/2 \\ s-k<q_1}} \frac{q_2}{q_1} B_{sk} = \varphi_2(q,C) \; .$$

Thus, with this nondecreasing and symmetric cost function, the agent who demands more always pays less. This violates Ranking.

Example 1.2.2. *Ranking and the Shapley-Shubik method*

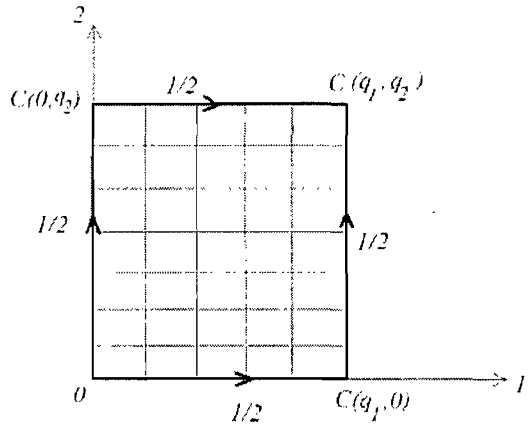
The Shapley-Shubik CSM is defined as the average of all the methods based on extremal paths, see Figure 1.2 for illustration. Unlike the Aumann-Shapley method, Shapley-Shubik pricing meets Ranking. We provide a simple argument in the case of two agents. Indeed, for any cost function C in $\mathscr{C}(N)$ (recall that $C(0,0) = 0$), the Shapley-Shubik shares are given by :

$$\varphi_1(q,C) = \frac{1}{2}[C(q_1,q_2) - C(0,q_2)] + \frac{1}{2}C(q_1,0),$$
$$\varphi_2(q,C) = \frac{1}{2}[C(q_1,q_2) - C(q_1,0)] + \frac{1}{2}C(0,q_2).$$

It follows that $\varphi_2(q,C) - \varphi_1(q,C) = C(0,q_2) - C(q_1,0)$. Hence, assuming that C is symmetric and $q_1 \leq q_2$, we have $C(0,q_2) = C(q_2,0) \geq C(q_1,0)$; the inequality stemming from the fact that C is nondecreasing. Therefore, $\varphi_2(q,C) \geq \varphi_1(q,C)$ and Ranking is satisfied.

The following section studies the implications of this axiom for the flow. In the sequel, unless otherwise specified, we consider a *fixed* profile q satisfying the conditions : $q \neq 0_n$ and $0 \leq q_1 \leq q_2 \leq .. \leq q_n$, i.e. the agents' demands are ranked from the smallest one to the highest one, with the latter being positive.

FIG. *1.2 – The Shapley-Shubik shares*



agent 1 pays $\frac{1}{2}C(q_1,0)+\frac{1}{2}[C(q_1,q_2)-C(0,q_2)]$

agent 2 pays $\frac{1}{2}C(0,q_2)+\frac{1}{2}[C(q_1,q_2)-C(q_1,0)]$

1.3 Ranking and the flow

1.3.1 The case of two agents

Let us start by examining the case of two agents ($n=2$). The results give an idea of what to expect in the general model ($n \in \mathbb{N}$). In addition, as we shall see further on, the case $n=2$ is specific because not only we derive implications for the flow system, but we are able to characterize the Ranking axiom too.

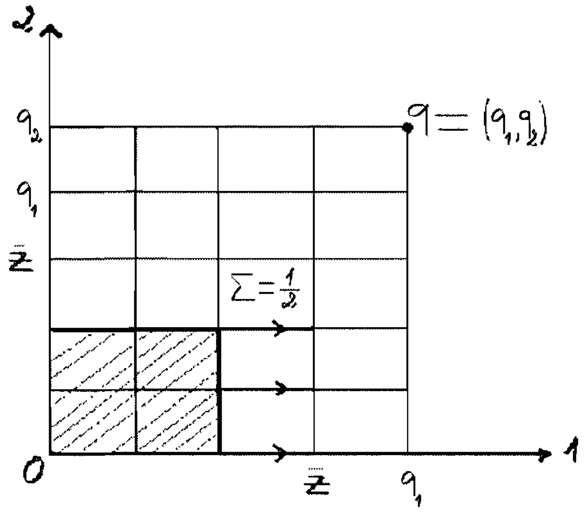
Fix $q \in [(0,0),(\bar{q},\bar{q})]$, with $q_1 \leq q_2$, and φ a CSM satisfying Ranking. Since there is no possible confusion (q is fixed), we abuse notation and write f instead of f^q, when referring to the corresponding flow to q. All the proofs of this Section are relegated to the Appendix.

Lemma 1.3.1. *If* $0<\bar{z}\leq q_1$, *then*

$$\sum_{z_2\in[0,\bar{z}-1]} f_1(\bar{z},z_2)=\frac{1}{2}=\sum_{z_1\in[0,\bar{z}-1]} f_2(z_1,\bar{z})\,. \tag{1.6}$$

As illustrated in Figure 1.3, Ranking implies that the sum of all horizontal (resp. ver-

FIG. 1.3 – Ilustration of Lemma 1.3.1



tical) flows exiting from any square originating from 0 and contained in $[(0,0);(q_1,q_1)]$ is equal to $\frac{1}{2}$.

Notice that these results have a flavor of symmetry, even though they are expressed in terms of sums of edges of the flow. The following result is a refinement of the previous lemma, since it claims the symmetry for single edges of the flow. In addition, we show that this symmetry property is sufficient to guarantee the Ranking property.

Proposition 1.3.1. ***Characterization result*** $(n = 2)$
Let $q = (q_1, q_2) \in [(0,0), (\overline{q}, ..\overline{q})]$ *with* $q_1 \leq q_2$. *A CSM represented by the flow* f *meets Ranking iff :* f *is* symmetric *inside the square* $[(0,0);(q_1,q_1)]$, *i.e.*

$$\forall\, z = (z_1, z_2) \in [(0,0);(q_1, q_1 - 1)],^{4} \quad f_1(z_1, z_2) = f_2(z_2, z_1). \tag{1.7}$$

The previous result describes the set of all discrete and additive CSMs satisfying Ranking. Since it is not always the case that $q_1 \leq q_2$ (as we have assumed in Lemma

[4] *This might look like an asymmetry but is not, because* $(z_1, z_2) \in [(0,0);(q_1, q_1 - 1)]$ *if and only if* $(z_2, z_1) \in [(0,0);(q_1 - 1, q_1)]$. *As shown in Figure 1.5-(a), the flow is symmetric inside the square* $[0,(q_1,q_1)]$, except for the North-East border of the square.

1.3.1), we have to consider the minimum of the two demands q_1 and q_2. Thus, our property is satisfied if and only if the flow is symmetric inside the square defined by the smallest coordinate.

Ranking requires that symmetric edges be crossed by the same flow (see Figure 1.5-(a)). Unlike Lemma 1.3.1, it is not possible to further refine this condition, since the symmetry is expressed here in the simplest possible way.

Remark 1.3.1. Apart from the symmetry requirement, there are some implicit conditions concerning the flow outside that square. Indeed, for Ranking to be satisfied, we need to have :

$$f_2(q_1,z_2) = \sum_{z'=q_1,..,q_2} f_1(z_2,z'); \ \forall \ z_2 = 0,..,q_1 \ . \tag{1.8}$$

But this would be redundant in the statement of Proposition 1.3.1 because it is implied by the symmetry inside the square and the flow conservation property. However, as we shall see further on, condition (1.8) is no longer redundant in higher dimensions ($n \geq 3$).

1.3.2 The general case : $n \in \mathbb{N}$ with $n \geq 3$

In the model with three agents or more, let us start with two preparatory statements which are similar to Lemma 1.3.1. These results are useful in proving the main finding of the paper.

Let $q \in [0_n, (\bar{q},..,\bar{q})]$ be such that $0 \leq q_1 \leq ... \leq q_n$. Consider a CSM φ satisfying Ranking, Again since q is fixed it reveals convenient to denote by f (instead of f^q) the corresponding flow to q.

Lemma 1.3.2. *Fix $i,j \in \{1,..,n\}$ with $i < j$. If $0 < \bar{z} \leq q_i = min\ \{q_i,\ q_j\}$, then we have :*

$$\sum_{z_j \in [0,\bar{z}-1]} \ \sum_{z_{-ij} \in [0_{-ij};q_{-ij}]} f_i(z_i = \bar{z}, z_j, z_{-ij}) = \frac{1}{2}\,, \tag{1.9}$$

where $z_{-ij} \in \mathbb{N}^{n-2}$ is obtained by suppressing the i-th and the j-th coordinates of $z \in \mathbb{N}^n$.

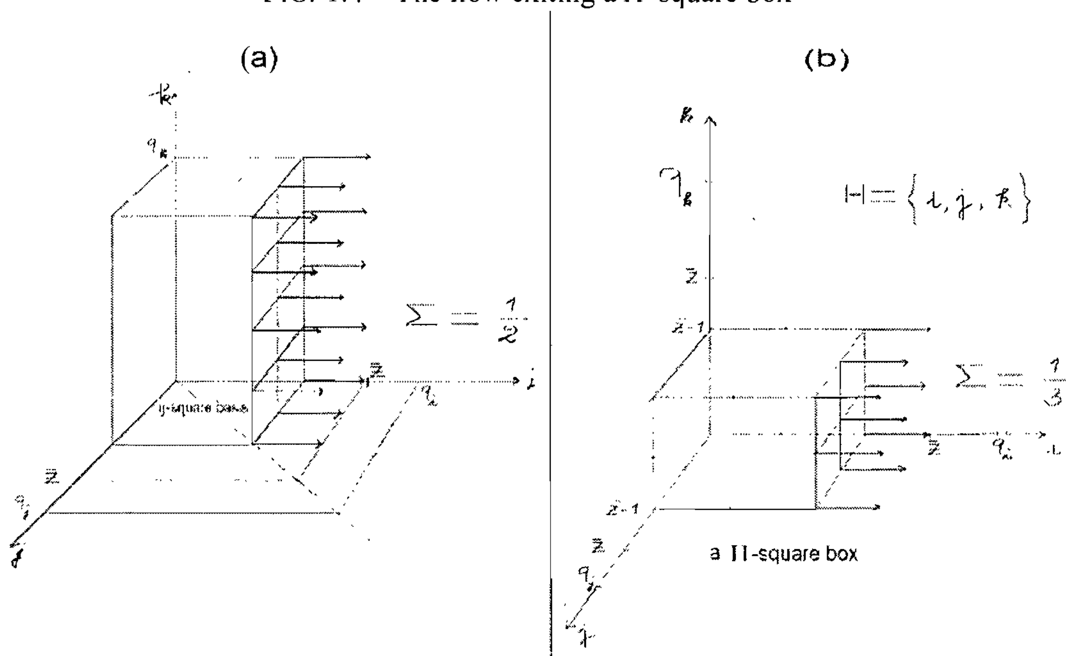
As illustrated in Figure 1.4-(a), the previous lemma states that for any ij-square box (a set of the type $[0_n;(\underbrace{\bar{z}-1}_{i},\underbrace{\bar{z}-1}_{j},q_{-ij})]$), the sum of all i-oriented outgoing flows is equal to $\frac{1}{2}$. Notice that the same holds for direction j. This property is the generalization of Lemma 1.3.1 to the case of three agents or more. Yet, it is not enough in order to prove the main finding of the paper. We have to refine it further, which is the object of the next lemma.

Lemma 1.3.3. *Let $H \subseteq N$, with $|H| = h \geq 2$. If $0 < \bar{z} \leq min\{q_l,\ l \in H\}$, then we have :*

$$\sum_{z_{H\setminus i} \in \left[0_{H\setminus i};(\bar{z}-1)1_{H\setminus i}\right]} \ \sum_{z_{N\setminus H} \in [0_{N\setminus H};q_{N\setminus H}]} f_i(z_i=\bar{z},z_{H\setminus i},z_{N\setminus H}) = \frac{1}{h} \quad \text{for all } i \in H, \tag{1.10}$$

where $z_A \in \mathbb{N}^A$ obtains from $z \in \mathbb{N}^N$ by considering only coordinates of which indexes are in A and $1_A = (1,..,1) \in \mathbb{N}^A$, for any $A \subseteq N$.

FIG. 1.4 – The flow exiting a H-square box



Call a set of the type $[0_n;(\bar{z}-1)1_{H\setminus i},q_{N\setminus H})]$ a H-square box. As shown in Figure

1.4-(b), if one considers a H-square box, then for any direction $i \in H$, the i-oriented outgoing flows sum up to the same constant, which is $\frac{1}{h}$. Notice that the result of Lemma 1.3.2 obtains by taking $h = 2$.

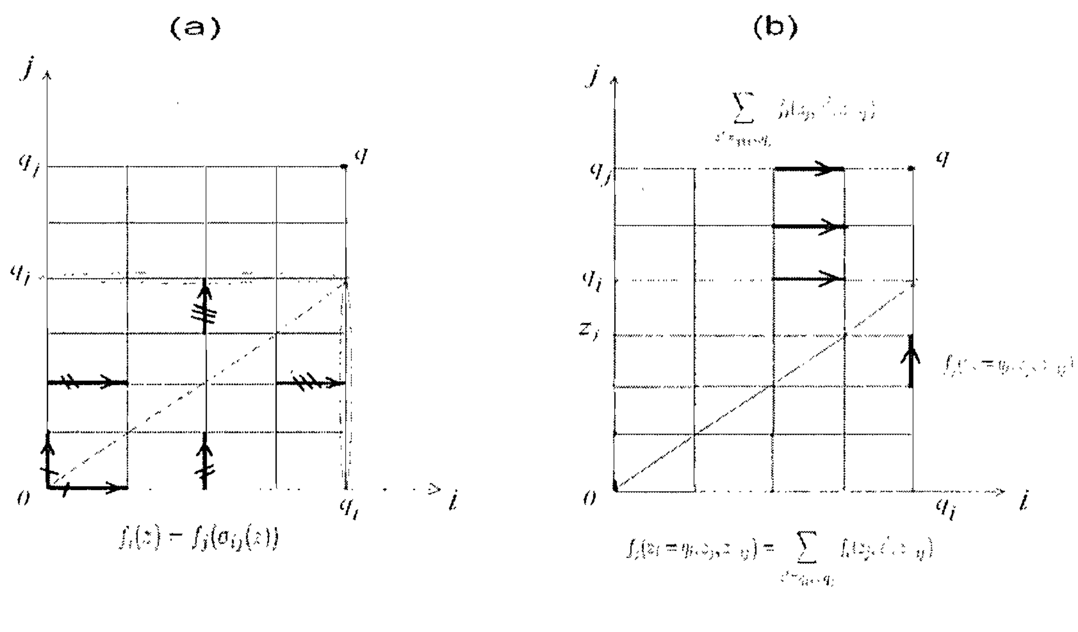
Building on the previous lemmas, one obtains (as in the case of two agents) symmetry conditions that are expressed on single edges of the flow.

Theorem 1.3.1. *Symmetry of the flow*
Let $q \in [0_n, (\overline{q}, .., \overline{q})]$ *such that* $0 \leq q_1 \leq \ldots \leq q_n$. *If a CSM represented by the flow f satisfies Ranking, then* $\forall\ i, j \in N$ *(with* $i < j$*)*, $\forall\ z \in [0_n; q]$ *such that* $(z_i, z_j) \in [(0,0); (q_i, q_i - 1)]$, *we have :*

$$\diamond \quad f_i(z) = f_j(\sigma_{ij}(z))\ , \tag{1.11}$$

$$\diamond \quad f_j(q_i, z_j, z_{-ij}) = \sum_{z'=q_i,..,q_j} f_i(z_j, z', z_{-ij}),\ \forall\ z_j = 0, .., q_i. \tag{1.12}$$

FIG. 1.5 – Symmetry of the flow



The conditions of Theorem 1.3.1 generalize the property obtained in the model with two agents. They are expressed in a compact form and are easily verifiable, since it is pretty straightforward to determine if a given flow is symmetric. However, as shown by the following example, these conditions are not sufficient to characterize Ranking in the model with three agents or more.

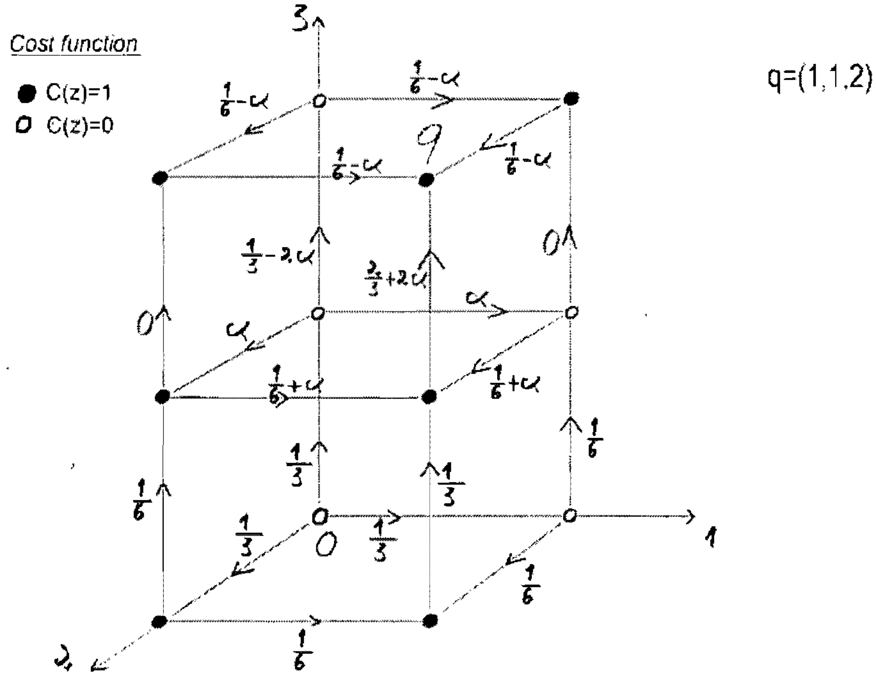
Remark 1.3.2.

- The symmetry of the flow extends beyond the cube defined by the lowest coordinate. Actually, it is a pairwise symmetry condition : in each each slice (i.e. for any fixed z_{-ij}), the flow should look like Figure 1.5-(a).
- Unlike the case with two agents, it is necessary to specify what happens in each slice outside the square defined by the lowest coordinate; see Figure 1.5-(b). Recall that this condition was redundant in the case of two agents.

Example 1.3.1. *Conditions of theorem 1.3.1 are not sufficient*
In the model with three agents, let us consider the flow presented in Figure 1.6 with $0 \leq \alpha < \frac{1}{6}$. The reader can check that the considered flow satisfies the symmetry properties of Theorem 1.3.1. For instance, we have : $f_1(1,0,0) = f_2(0,1,0) = f_3(0,0,1) = \frac{1}{3}$, $f_1(1,0,2) = f_2(0,1,2) = \frac{1}{6} - \alpha$ and $f_3(0,1,1) = \frac{1}{6} = \alpha + [\frac{1}{6} - \alpha] = f_2(0,1,1) + f_2(0,1,2)$ (illustration of condition (1.12)).
Figure 1.6 depicts the restriction on $[0,q]$ of the 0-1 cost function defined by $C(z) = 1$ if and only if $[(z_2 > 0)$ or $(z_2 = 0$ and $z_1 + z_3 \geq 3)]$, which is 13-symmetric. And though q_3 is greater than q_1, agent 3 pays nothing while agent 1 pays a positive cost. This violation of Ranking is presented here with three agents, but it could easily be extended to higher dimensions.

Hence, our symmetry conditions do not characterize Ranking in the model with three agents or more. However, conditions (1.11) and (1.12), taken together, guarantee Equal Treatment of Equals (ETE), a consequence of Ranking. Indeed, suppose that $q \in [0_n, (\overline{q}, .., \overline{q})]$ with $q_i = q_j$. Together, conditions (1.11) and (1.12) are equivalent to :

FIG. 1.6 – Symmetry conditions are not sufficient



$$0 \leq \alpha < \tfrac{1}{6},\ q = (1,1,2) \text{ and } \varphi_1(q,C) = \tfrac{1}{6} - \alpha,\ \varphi_2(q,C) = \tfrac{5}{6} + \alpha,\ \varphi_3(q,C) = 0.$$

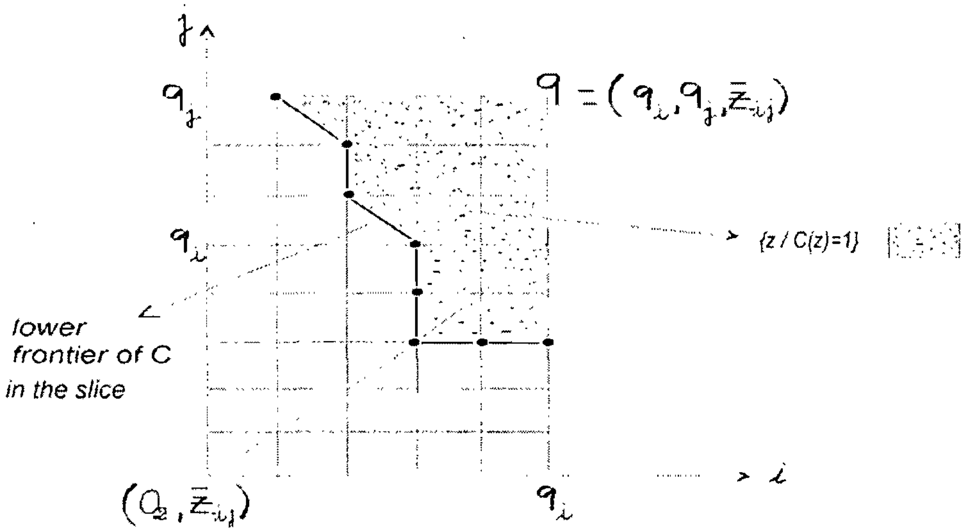
$f_i(z) = f_j(\sigma_{ij}(z))$, for all $z \in [0_n, q]$. Hence, for any ij-symmetric cost function C, computing the cost shares according to equation (1.5) will result in equal shares for agents i and j.

1.4 Ranking for elementary fixed flows

In this section, we investigate the implications of Ranking beyond the symmetry of the flow and for the specific case of elementary fixed flows, we propose a characterization result. First, let us introduce some notation.

Suppose that C is a 0-1 cost function. For any $\bar{z}_{-ij} \in [0_{-ij}, q_{-ij}]$, consider the set $X^C = \{z \in [0_n, (\bar{q}, .., \bar{q})] \text{ s.t. } C(z) = 1\}$. We call the *lower frontier of the cost function C in the slice* $z_{-ij} = \bar{z}_{-ij}$ the set defined by $\underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C) = \{z \in X^C \text{ s.t. } z_{-ij} = \bar{z}_{-ij} \text{ and } \{z - e_i, z - e_j\} \not\subseteq X^C\}$; see Figure 1.7 for illustration. Notice that a set F is admissible as the

FIG. 1.7 – Lower frontier of C



lower frontier (in the slice $z_{-ij} = \bar{z}_{-ij}$) of an $i-j$ symmetric $0-1$ cost function, if it is a *non-increasing* line of which the intersection with the set $[(0_{ij}, \bar{z}_{-ij}); (q_i, q_i, \bar{z}_{-ij})]$ is $i-j$ symmetric i.e. if the set $F \bigcap [(0_{ij}, \bar{z}_{-ij}); (q_i, q_i, \bar{z}_{-ij})]$ is invariant under the transformation σ_{ij}. Such a set F shall be called a *symmetric lower frontier*. Unlike Figure 1.8-(b), Figure 1.8-(c) depicts a symmetric lower frontier in the slice $z_{-ij} = \bar{z}_{-ij}$.

Using the flow and the concept of symmetric lower frontier, one can characterize the Ranking axiom in the following way.

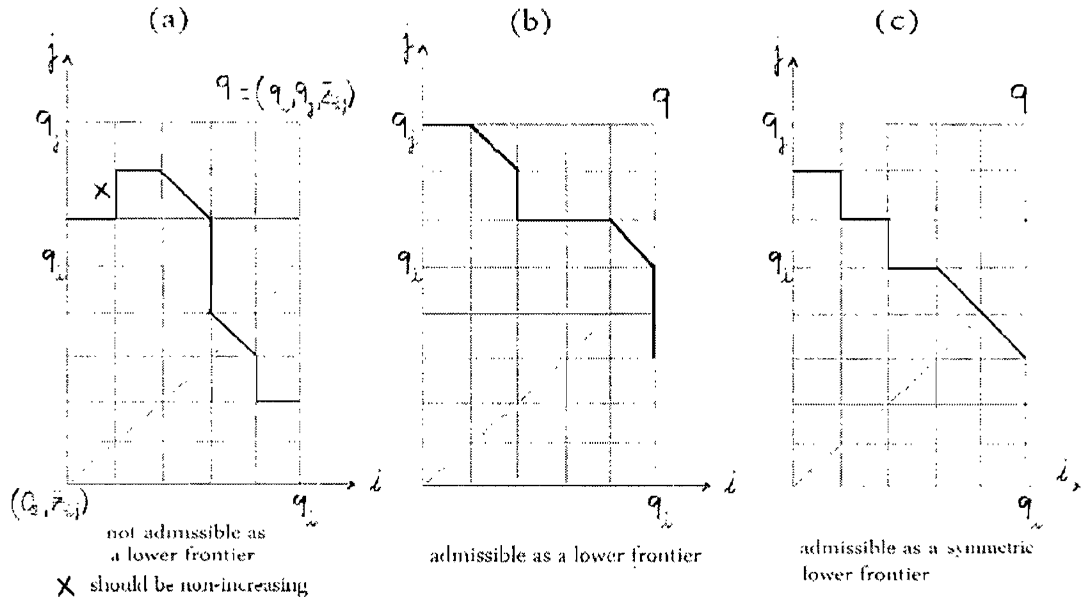
Proposition 1.4.1. *Let* $q = (q_1, .., q_2) \in [0_n, (\bar{q}, .., \bar{q})]$ *with* $q_1 \leq q_2 \leq .. \leq q_n$. *A CSM represented by the flow f satisfies Ranking iff :* $\forall\, i, j \in N$ *(with* $i < j$*),* $\forall\, \bar{z}_{-ij} \in [0_{-ij}, q_{-ij}]$, *we have :*

$$\sum_{z \in F_{\bar{z}_{-ij}}} f_i(z) \leq \sum_{z \in F_{\bar{z}_{-ij}}} f_j(z)\ , \tag{1.13}$$

for any symmetric lower frontier $F_{\bar{z}_{-ij}}$.

Proof. $(\Rightarrow)$ Let $F_{\bar{z}_{-ij}}$ be a lower frontier in the slice $z_{-ij} = \bar{z}_{-ij}$ (with $i < j$) and

FIG. 1.8 – (Counter)Examples of lower frontiers



consider the cost function $C_{F_{\bar{z}_{-ij}}}$ defined as follows :

$$C_{F_{\bar{z}_{-ij}}} = \begin{cases} 1 & if\ z_l > \bar{z}_l \text{ for some } l \in N \setminus \{i,j\}, \\ 1 & if\ \ (z_{-ij} = \bar{z}_{-ij}) \text{ and } (\exists\, z' \in F_{\bar{z}_{-ij}} \text{ s.t. } z' \leq z), \\ 0 & \text{otherwise.} \end{cases}$$

Note that by construction, $F_{\bar{z}_{-ij}}$ is precisely the lower frontier of the cost function $C_{F_{\bar{z}_{-ij}}}$ in the slice $z_{-ij} = \bar{z}_{-ij}$. Since $C_{F_{\bar{z}_{-ij}}}$ is an $i-j$ symmetric cost function, Ranking requires that the shares $\varphi_i(C_{F_{\bar{z}_{-ij}}})$ and $\varphi_j(C_{F_{\bar{z}_{-ij}}})$ satisfy :

$$\varphi_i(C_{F_{\bar{z}_{-ij}}}) = \sum_{z \in F_{\bar{z}_{-ij}}} f_i(z) \leq \sum_{z \in F_{\bar{z}_{-ij}}} f_j(z) = \varphi_j(C_{F_{\bar{z}_{-ij}}}).$$

Observe that the above equalities hold because all variations of $C_{F_{\bar{z}_{-ij}}}$ in directions i and j occur in the slice $z = \bar{z}_{-ij}$.

($\Leftarrow$) Suppose that (13) is true for any lower frontier $F_{\bar{z}_{-ij}}$ (for any $\bar{z}_{-ij} \in [0_{-ij}, q_{-ij}]$ and for any $i < j$). If C is an $i-j$ symmetric cost function taking the values 0 or 1, then applying (1.5) yields :

$$\varphi_i(C) = \sum_{\bar{z}_{-ij} \in [0_{-ij}, q_{-ij}]} \ \sum_{z \in \underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)} f_i(z) \leq \sum_{\bar{z}_{-ij} \in [0_{-ij}, q_{-ij}]} \ \sum_{z \in \underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)} f_j(z) = \varphi_j(C).$$

Finally, using the decomposition (A.12), we can claim that the desired inequality is satisfied for any $i-j$ symmetric cost function. ■

Although Proposition 1.4.1 completely characterizes Ranking, it is not fully satisfactory because property (1.13) lacks conciseness and is therefore hardly verifiable. Indeed, there are a large number of symmetric lower frontiers and one needs to check (1.13) for each one of them in order to determine whether a flow method satisfies the axiom. However, for the so-called elementary fixed flows, this property leads to an operational characterization result.

In the sequel, we vary the demand profile $q \in [0_n, (\overline{q},..,\overline{q})] \subset \mathbf{IN}^n$ (it is no longer considered fixed).[5] Consider the class of fixed-flow methods.[6] Given f^q, the flow to q, the flow to any lower profile q' obtains by projecting f^q on the box $[0_n, q']$.[7] In particular, the flow $f^{(\overline{q},..,\overline{q})}$ to the highest profile completely defines the CSM. Since there is no ambiguity, we will identify the fixed flow method f with $f^{(\overline{q},..,\overline{q})}$.

Definition 1.4.1. *Elementary fixed flows*

◇ Let an *increasing path* from 0_n to q be a sequence of the type $\Gamma = (\Gamma_k,\ k = 0, 1, .., \sum_{s=1}^{n} q_s)$ wich satisfies : $\forall\, k = 1, .., \sum_{s=1}^{n} q_s,\ \Gamma_k = \Gamma_{k-1} + e_i$, for some $i \in N(\Gamma_{k-1}, q)$, with $\Gamma_0 = 0_n$ and $\Gamma_{\sum_{s=1}^{n} q_s} = q$.

◇ We call a symmetric fixed flow *elementary* (*elsff*) if its support is minimal, i.e. if it is composed of the $n!$ permutations of an increasing path going from 0_n to $(\overline{q},..,\overline{q})$.

Figure 1.9 depicts an *elsff*. We derive a necessary and sufficient condition for an elementary fixed flow to meet Ranking.

Consider the following property : for any $i,j \in N$ and for any fixed $\overline{z}_{-ij}$, the flow in the slice $z_{-ij} = \overline{z}_{-ij}$ is such that

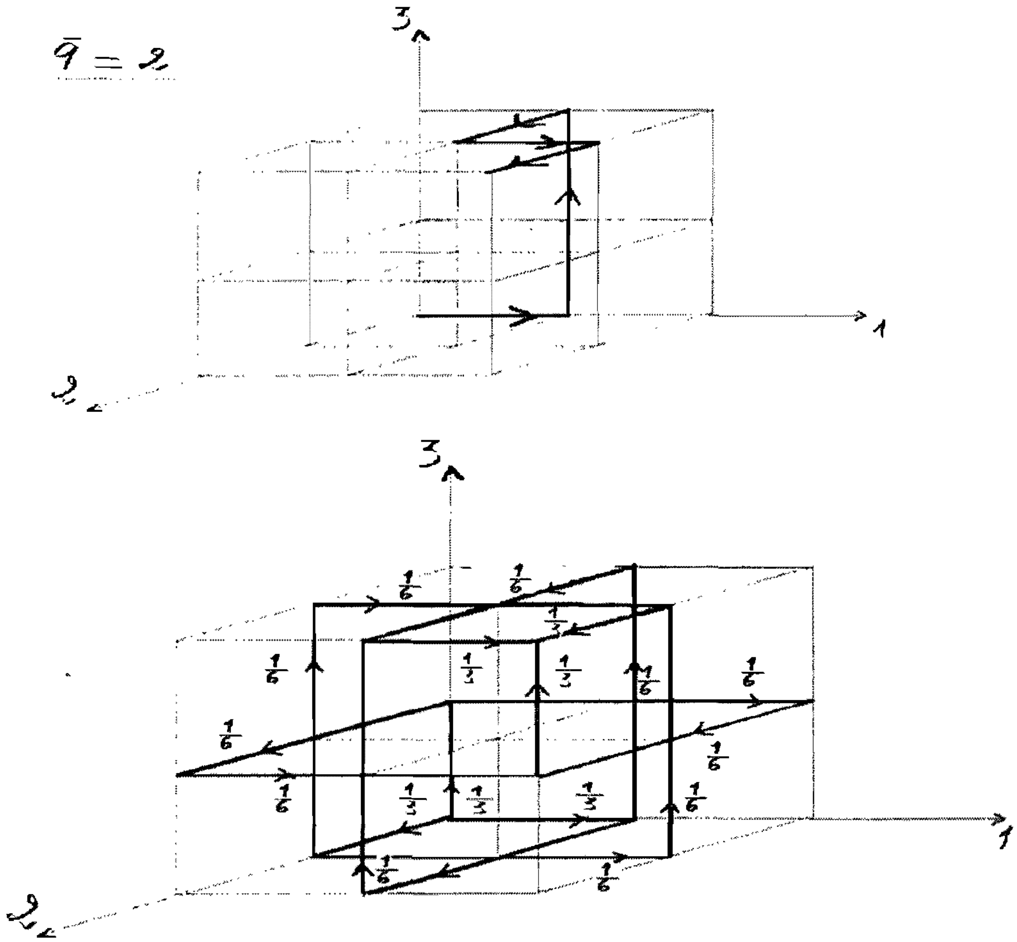
$$(z_j < z_i \text{ and } f_j(z_i,z_j,\overline{z}_{-ij}) > 0) \Rightarrow f_j(z_i+k,z_j+1,\overline{z}_{-ij}) \geq f_j(z_i,z_j,\overline{z}_{-ij}) \text{ for some } k \geq 0. \tag{1.14}$$

[5] For convenience, we assume that $\overline{q}$ is finite ; all demands are, therefore, bounded.

[6] Among the set of flow methods, they are the only ones satisfying the property *Irrelevance of Dummy Changes* (a stronger version of dummy), the interested reader can see Sprumont (2008).

[7] See Moulin and Sprumont (2005) for a formal definition of fixed flows.

FIG. 1.9 – An increasing path and all its permutations defining an *elsff*



Condition (1.14) says that *in each slice*, if the flow gets closer to the diagonal, then it must keep on doing so (until it reaches the diagonal) at least with the same intensity.

Proposition 1.4.2. Characterization result for *elsff*s

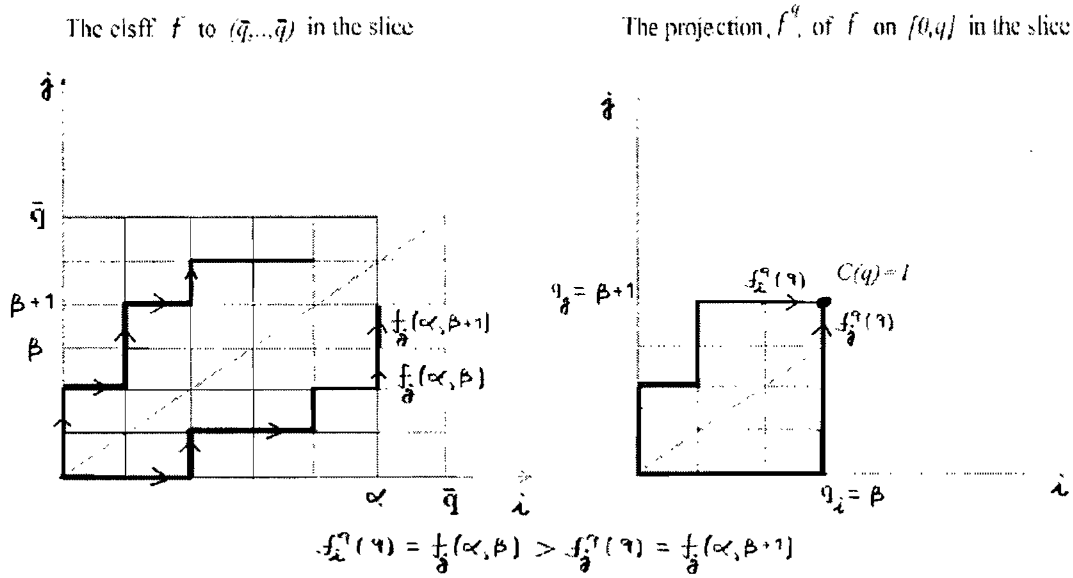
An elementary fixed flow f meets Ranking iff it satisfies condition (1.14).

Proof. ($\Rightarrow$) By way of contradiction, suppose that for some $i, j \in N$, $\exists\ \alpha, \beta \in [0, \bar{q}]$ (with $\beta < \alpha$), $\bar{z}_{-ij} \in [0_{n-2}; (\bar{q}, .., \bar{q})]$ such that :

$$f_j(\alpha, \beta, \bar{z}_{-ij}) > 0 \text{ and } f_j(\alpha + k, \beta + 1, \bar{z}_{-ij}) < f_j(\alpha, \beta, \bar{z}_{-ij}) \text{ for any } k \geq 0. \quad (1.15)$$

Consider $q=(\beta,\beta+1,\bar{z}_{-ij})$ and the 0-1 cost function defined by $C(z)=1$ if and only if $[(z\geq q=(\beta,\beta+1,\bar{z}_{-ij})$ or $z\geq q'=(\beta+1,\beta,\bar{z}_{-ij})$ or $(z_t\geq \bar{z}_t)$ for some $t\in N\setminus ij]$. Since

FIG. 1.10 – Necessity of condition (1.14) for *elsff*s



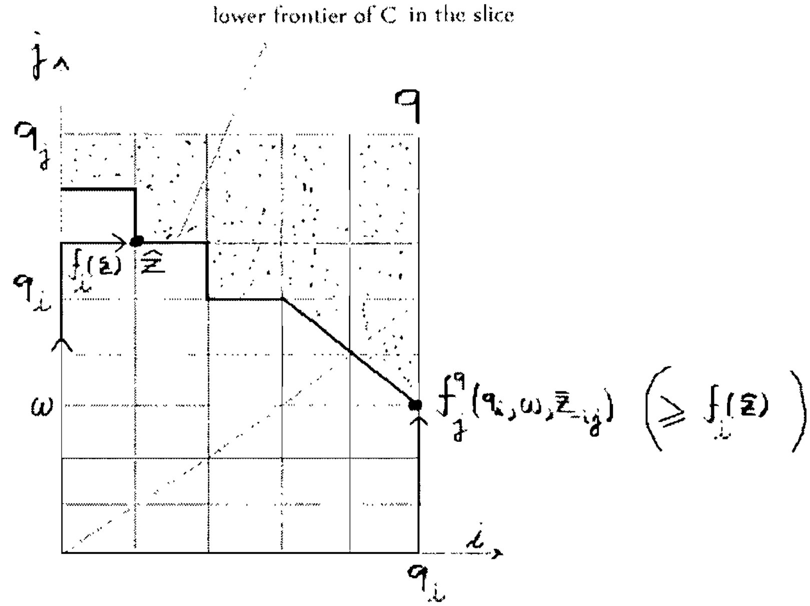
f is an elementary fixed flow, we can write $\varphi_i(q,C)=f_i^q(\beta,\beta+1,\bar{z}_{-ij})=f_i(\beta,\alpha,\bar{z}_{-ij})$ (see Figure 1.10) and $\varphi_j(q,C)=f_j^q(\beta,\beta+1,\bar{z}_{-ij})=f_j(\alpha+k,\beta+1,\bar{z}_{-ij})$, for some $k\geq 0$. It follows from (1.15) that $\varphi_j(q,C)<\varphi_i(q,C)$, although $q_i=\beta<q_j=\beta+1$. This contradicts Ranking.

($\Leftarrow$) Suppose that (1.14) is satisfied and fix $q\in[0_n;(\bar{q},..,\bar{q})]$ such that $q_i\leq q_j$. For any $i-j$ symmetric cost function C and its lower frontier $\underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)$ in the slice $z_{-ij}=\bar{z}_{-ij}$, since f is an *elsff*, there exists at most one $\hat{z}\in\underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)$ such that $f_i^q(\hat{z})>0$. Suppose that $\hat{z}$ exists with $\hat{z}\in[(0,q_i,\bar{z}_{-ij});(q_i,q_j,\bar{z}_{-ij})$;[8] For an *elsff*, we have $\sum_{z\in\underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)} f_i^q(z)=f_i^q(\hat{z})=f_j^q(\hat{z}_j,\hat{z}_i,\hat{z}_{-ij})$. Given that C is symmetric and nondecreasing, there exists a

[8] We rule out the case $\hat{z}\in[(0_2,\bar{z}_{-ij});(q_i,q_i-1,\bar{z}_{-ij})]$, since the symmetry of the flow easily yields the desired result with equality in this case.

unique $\omega \in [\hat{z}_i, q_2]$ such that $(q_1, \omega, \hat{z}_{-ij}) \in \underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)$. See Figure 1.11.[9] Applying (1.14)

FIG. 1.11 – Sufficiency of condition (1.14) for *elsff*s



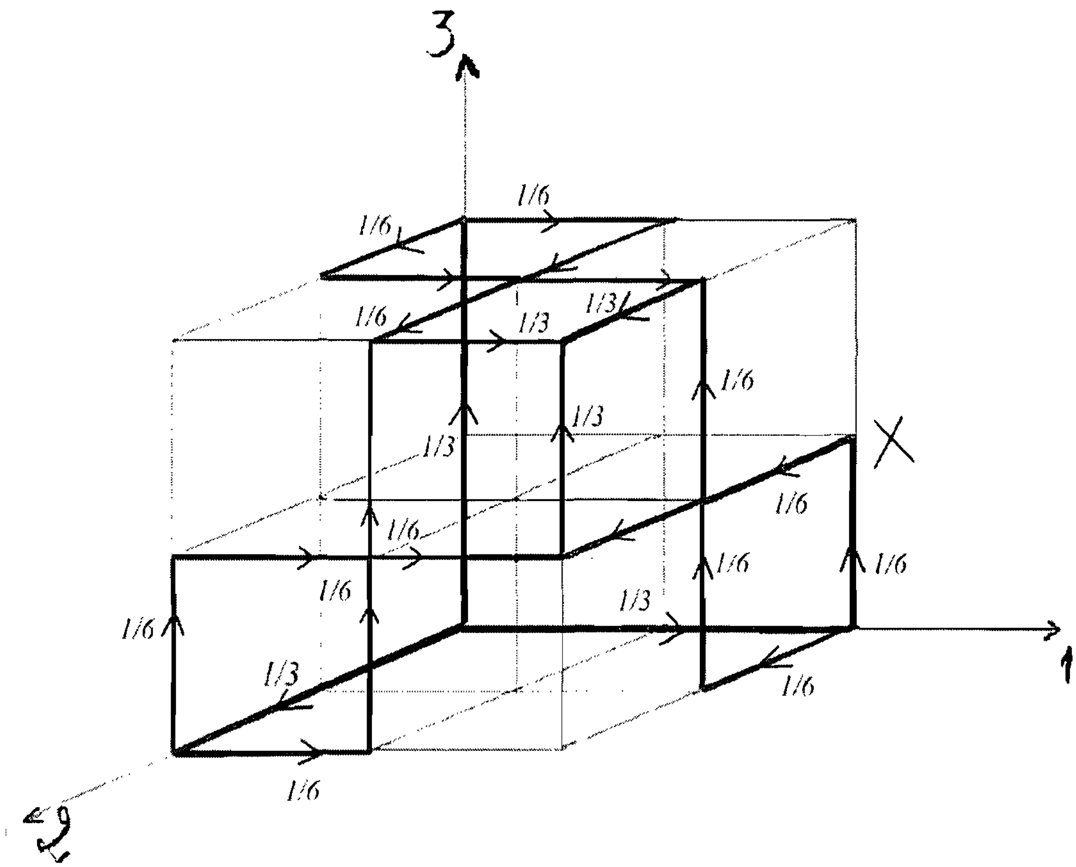
-repeatedly, possibly-, one can write : $f_j^q(\hat{z}_j, \hat{z}_i, \hat{z}_{-ij}) \leq f_j^q(q_1, \omega, \hat{z}_{-ij}) = \sum_{z \in \underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C)} f_j^q(z)$,
which proves Ranking by Proposition 1.4.1. ■

The *elsff* $\hat{f}$ described by Figure 1.12 violates (1.14). Indeed, in the slice $z_2 = 0$, we have $\hat{f}_3(2,0,1) = \frac{1}{6} > \hat{f}_3(2,0,2) = 0$. Hence, the CSM represented by this *elsff* does not meet Ranking.

Notice that Serial cost sharing (see Figure 1.13-(a)) and the Shapley-Shubik CSM (see Figure 1.13-(b)) are represented by elementary symmetric fixed flows meeting condition (1.14); henceforth, they both satisfy Ranking. For these two methods, the flow in each slice is "closed", i.e. it does return to the diagonal. Some other *elsff*s satisfy (1.14) without exhibiting this property. For instance, observe that the *elsff* $\tilde{f}$ depicted by Figure 1.14-(a) meets (1.14), although it is not "closed" : it does not return to the diagonal in

[9] We assume that C(q)=1, otherwise the result is trivial. Also notice that we have ruled out the case where $\underline{\mathrm{Fr}}_{\bar{z}_{-ij}}(C) \subseteq \{z \text{ s.t. } z_i + z_j \leq q_i\} \cap \{z \text{ s.t. } z_{-ij} = \bar{z}_{-ij}\}$; in this case, indeed, the symmetry of the fixed flow would yield equal contributions to the shares of agents i and j for that slice.

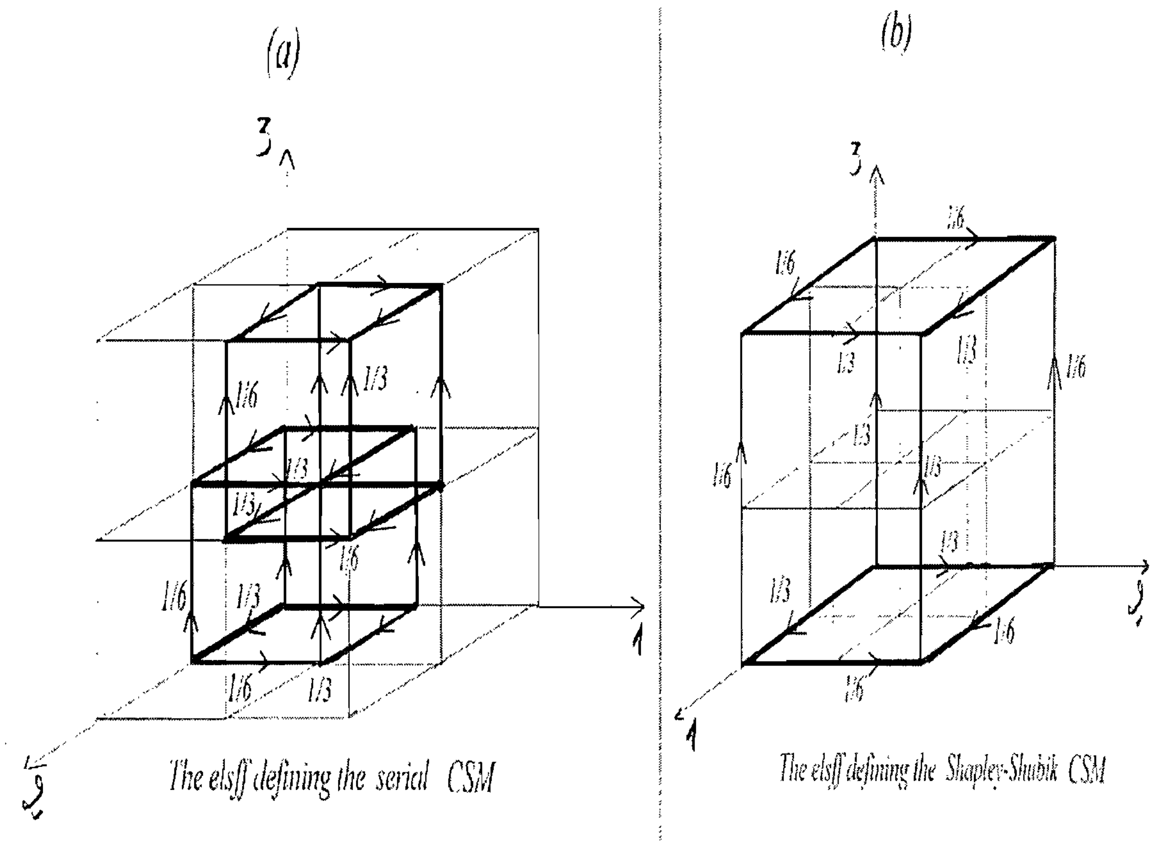
FIG. 1.12 – The *elsff* $\hat{f}$ violates condition (1.14), $\bar{q} = 2$



the slice $z_2 = 0$. See Figure 1.14-(b).

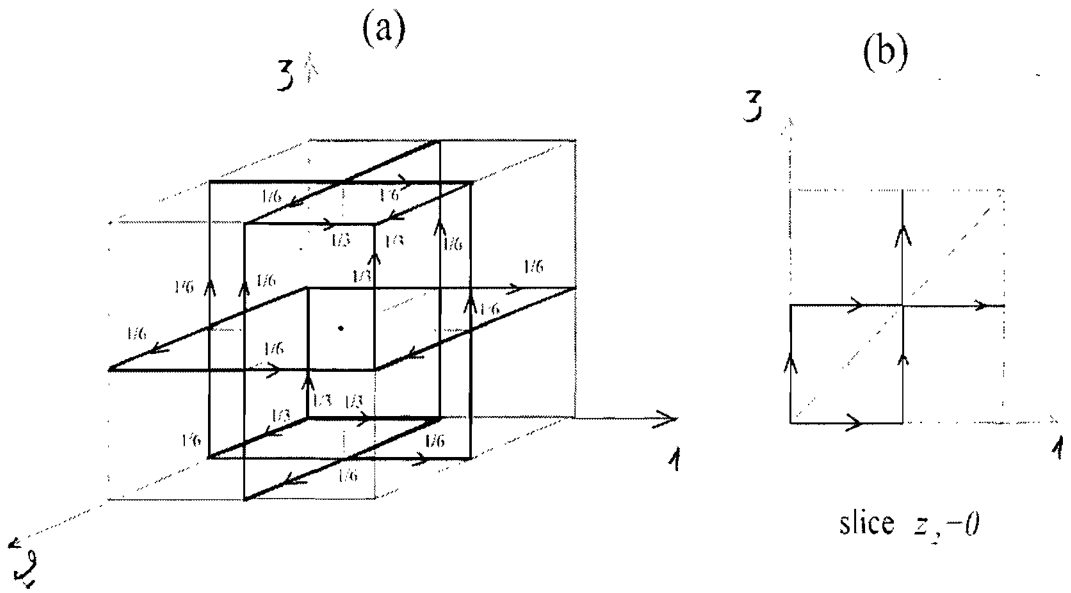
Finally, note that for the wider class of symmetric fixed-flows, the requirement (1.14) would be sufficient but not necessary for Ranking. For example, consider the symmetric fixed flow defined as the average of $\hat{f}$ (Figure 1.12) and $\tilde{f}$ (Figure 1.14), $\check{f} = 1/2(\hat{f} + \tilde{f})$. Condition (1.14) is not satisfied by $\check{f}$: in the slice $z_2 = 0$, we have $\check{f}_3(2,0,1) = \frac{1}{12} > \check{f}_3(2,0,2) = 0$. However, $\check{f}$ does satisfy Ranking. Indeed, it is tedious but straightforward to show that $\check{f}$ satisfies condition (1.13), for any $q \in [0_3, (2,2,2)]$. More generally, any strict convex combination of $\hat{f}$ and $\tilde{f}$ violates (1.14). On the other hand, it satisfies

FIG. 1.13 – The *elsff*s representing the Serial and Shapley-Shubik CSMs for $\bar{q} = 2$



Ranking if and only it gives to $\tilde{f}$ a weight at least equal to half.

FIG. 1.14 – $\tilde{f}$ meets (1.14) and is not "closed"



1.5 Concluding Remarks

The paper studies the Ranking property for the discrete model, where demands are integer-valued. As a recap of the main findings, recall that the symmetry of the flow *inside* the square defined by the smallest demand entirely characterizes Ranking in the case of two agents. Henceforth, (for symmetric cost functions) a CSM always ranks the cost shares of the two agents accordingly with the Ranking of their demands if and only if it is represented by a symmetric flow.

In the general framework ($n \geq 3$), we obtain a pairwise symmetry property (on the flow) which extends beyond the cube defined by the lowest demand and which generalizes the result obtained with two agents. The interpretation of this property is the following : for Ranking to be satisfied, symmetric paths which are used in the calculation of the cost shares should be given equal weights. However, with three agents or more, the symmetry of the flow is not sufficient to guarantee (1.13), which characterizes

Ranking.

Among the class of elementary fixed flows, we show that Ranking is characterized by a specific condition : in each slice, if the flow approaches the diagonal, then it should continue to do so (with at least the same intensity). Both Serial cost sharing and the Shapley-Shubik method satisfy this requirement since they are represented by elementary fixed flows that return to the diagonal in each slice.

The results suggest that if demands are interpersonally comparable and we want the cost shares to have the same ranking as the demands, then we ought to choose among the set of flow methods described above. And though there is an infinity of such methods, the implications of Ranking dramatically reduce the number of flows among which we have to choose.

Some questions remain of interest for further research. Firstly, how do these results translate in the continuous model and what is the class of continuous CSMs satisfying Ranking? In addressing this issue, the result by Moulin and Friedman (1999), which characterizes the set of all continuous methods meeting additivity and dummy, will certainly be an good starting point.

Secondly, it would be interesting to combine Ranking with other axioms in order to come up with some characterization results or recommendations (i.e. further reducing the set of admissible mechanisms). For instance, in the continuous model, we conjecture that requiring Ranking and Ordinality[10] (or even the milder Scale Invariance[11] condition) would pick the Shapley-Shubik method among all additive and dummy CSMs.

[10]See Sprumont (1998).

[11]The cost shares should not depend on the units used to measure the demands, see for instance Friedman and Moulin (1999) or Friedman (2003).

CHAPITRE 2

OPTIMAL MANAGEMENT OF STRATEGIC RESERVES OF NONRENEWABLE NATURAL RESOURCES

2.1 Introduction

An increasing number of countries are holding Strategic Petroleum Reserves (SPR).[1] Following the two 1970s oil crises, many western economies engaged in stockpiling the resource as well as in developing alternative ways for energy production in order to lessen their dependence on oil. Those supply disruptions have also generated some interest for the optimal management of such vital exhaustible resources, the provision of which might be subject to some randomness. Important contributions include Dasgupta and Heal (1974), who showed how the possibility of a substitute source of energy (whose discovery date is uncertain) can be taken into account by adjusting the discount rate. Teisberg (1981) designed a dynamic programming model in order to determine the optimal strategy for U.S. energy acquisition and storage.

In March 2001 the International Energy Agency (IEA), as a recommendation to its members, asked each importing country to maintain reserves equivalent to at least 90 days of oil consumption.[2] The USA currently owns the largest reserve, which is equivalent to 60 days of imports. Even some net exporting IEA members such as Denmark and the U.K. have started to stockpile.

From standard economic theory of exhaustible resources, we know that it is not optimal to stockpile the resource in a deterministic framework.[3] But this changes as soon

[1]For an overview of the facts concerning the global strategic oil reserves, see for instance *http ://en.wikipedia.org/wiki/Global_strategic_petroleum_reserves* and the references cited therein. See also Bamberger (2008) for a discussion of issues related more specifically to the US strategic oil reserves.

[2]For a description of the IEA's emergency measures, the interested reader can consult "Overview of IEA Oil Emergency Procedures and Measures In IEA Member Countries" at : *http ://www.iea.org/Textbase/work/2002/beijing/kuolt2.pdf.*

[3]This is because if there is no uncertainty about the future availability of the supply source, the dis-

as we introduce uncertainty of supply. If an importing country is likely to suffer a trade disruption (an embargo for instance), it has no option but to store in order to avoid zero consumption during disruption periods. This paper aims at determining the optimal stockpiling plan from the importing country's perspective.

Some works in the literature (see for example Loury [1983]) have tackled the issue of trade disruptions for a storable good. The present paper addresses the problem of trade disruption in the specific case of an exhaustible resource (for the sake of concreteness, we refer to **oil** throughout the paper), taking into account the specific feature of the price path, which follows as a consequence of exhaustibility.

Although the management of SPR has regained interest due to the instability on the supply side, the recent literature on the subject is not very abundant. Two important works are useful building blocks for this paper.

The first one, by Hillman and Long (1983), determines optimal domestic reserves depletion for a country under threat of a single trade embargo, that lasts forever. They solve for the optimal pricing by domestic firms after the embargo and discuss the issues raised by market structure : perfect competition or firms with market power. Basically, the result of the paper is a conservationist depletion policy, due to the embargo threat. Unlike that of Hillman and Long, the present paper is interested in the pre-disruption stockpiling policy for a country with no domestic deposit.

In the second paper, Bergström, Loury and Persson (1985) address the issue of the stockpiling. They allow for an infinite sequence of disruptions interspersed with free trade regimes. The main result of their model is the existence of an optimal size of the reserves that the importing country wants to maintain (in a steady state) in order to hedge against disruptions. However, since they assume a constant price, one can question its relevance for nonrenewable resources.[4]

counted cost of acquiring the resource just when the future need occurs is less than the cost of acquiring it now and stockpiling to satisfy that future need. With a positive discount rate, this is so even if the cost of stockpiling is zero, but all the more so if it is positive.

[4]For a detailed discussion about the evolution of nonrenewable resources prices in the light of the

In the present paper, we consider the possibility of several disruptions and solve for the optimal management (both stockpiling and depletion) of the SPR. An additional feature of our model is the possibility for the importing country to invest in a backstop technology in order to lessen its dependence on oil.

In Section 2, I present the model. The optimal stockpiling policy is derived in Section 3. I show that there exists a target SPR path that the importing country wants to reach. The optimal SPR policy is thus to increase the precautionary reserves as fast as allowed by the country's budget until the building up stage is completed. From that moment on, the reserves decrease. Section 4 focuses on the development of an alternative source of energy as a replacement of oil. I allow the importing country to invest in research at each date, the moment at which a backstop is discovered being stochastic. Basically, we show that it is in the best interest of the country to undertake *R&D*. Moreover, the incentive to accelerate research is higher when the country is running short of strategic reserves. Finally, Section 5 concludes with some remarks about the implications of our main findings for the SPR management.

2.2 The Model

Our world economy consists of an oil producer, which we will call country A for convenience, and several importing countries. Let us consider an importing country, B, which is endowed with a flow of monetary revenue Z (assumed constant).

2.2.1 The importing country's budget constraint

Let us denote by $m(t)$ the quantity of oil that is imported at price $p(t)$ and by $X(t)$, the strategic reserves held by country B at time t. The importing country's oil consumption, $q(t)$, can be written as : $q(t) = -\dot{X}(t) + m(t)$. Its budget constraint in a free trade regime

Hotelling rule, see Gaudet (2007).

is then given by :

$$p(t)m(t) = p(t)(\dot{X}(t) + q(t)) = Z - z(t), \qquad (2.1)$$

where $z(t)$ is the remaining monetary budget after oil purchases ($z(t)$ might be viewed as the consumption of a composite good the price of which is normalized to unity). A key assumption made here is that the country is small enough not to influence the price determination on the international oil market by varying its oil imports. For simplicity, we will assume that the extraction cost in country A is zero and that the price follows the Hotelling rule of nonrenewable resource extraction (Hotelling [1931]). Hence, the price of oil grows at the rate of interest r : $p(t) = p(0)e^{rt}$.

If country B is suffering an embargo (i.e. $m(t) = 0$), then the budget constraint becomes $z(t) = Z$ and the depletion of its strategic reserves is simply given by $\dot{X}(t) = -q(t)$.

2.2.2 Preferences

As in Bergström, Loury and Persson (1985), let us assume that the importing country's instantaneous utility is represented by a quasi-linear utility function, that depends on the consumption of both oil (q) and a composite good (z) :[5]

$$U(q(t), z(t)) = u(q(t)) + z(t) \qquad (2.2)$$

with $u' > 0$, $u'' < 0$ and $\lim_{q \to 0} u'(q) = +\infty$. This instantaneous utility is discounted over time at the constant rate of interest r.

2.2.3 The state of the market

The transition from the free trade regime to the embargo and vice versa follows a stochastic process. Let $s(t)$ denote the state of the market at time t. If the market is

[5]With this representation, the marginal utility of income is constant and $u(q(t))$ can be seen as the monetary value of consuming the amount of energy $q(t)$.

"on" (free trade), then $s(t) = 1$; if the market is "off" (embargo), then $s(t) = 0$. Let τ_0^n (respectively τ_1^n) be the date at which the nth embargo happens (respectively free trade resumes for the nth time).[6] Assuming that $s(t)$ follows a stationary Markovian process,[7] the density functions of the durations of the regimes are given, respectively, by :

$$h_0(\tau_0^n - \tau_1^n) = \theta_0 e^{-\theta_0(\tau_0^n - \tau_1^n)} \ \text{[resp. } h_1(\tau_1^n - \tau_0^{n-1}) = \theta_1 e^{-\theta_1(\tau_1^n - \tau_0^{n-1})}]. \tag{2.3}$$

For any t, $\theta_0 dt$ (resp. $\theta_1 dt$) represents the probability that the next embargo (free trade regime) will happen in the interval $[t, t+dt]$, conditional on the fact that it has not happened yet at date t. Also, with this specification, the average waiting time for an embargo to happen (resp. for trade to resume) is $\frac{1}{\theta_0}$ (resp. $\frac{1}{\theta_1}$).

In the special case where only one embargo may occur that lasts forever, we will have $n = 1$ and $\tau_1^n = 0$, so that h_1 is redundant.

2.2.4 R&D

As in Dasgupta and Heal (1974), we allow the importing country to invest in R&D starting at date $t = 0$. As a result of this research, there is a positive probability $\theta_b(y)dt$ at each date $t \geq 0$ that a *backstop technology* will be discovered within $[t, t+dt]$; where y represents country B's research effort, at a cost $C(y)$. The random date at which the backstop is discovered is exponentially distributed with

$$h_b(\tau_b) = \theta_b(y) e^{-\theta_b(y)\tau_b}. \tag{2.4}$$

The occurrence of the backstop makes energy available to country B at a constant marginal cost. I initially ignore the effort and the cost of investing in research (in the following

[6] At date $t = 0$, the ongoing regime is "free trade". Thus, we have the sequence :
$0 = \tau_1^1 < \tau_0^1 < \tau_1^2 < \tau_0^2 < ... < \tau_0^{n-1} < \tau_1^n < \tau_0^n < ...$

[7] The stationarity assumption implies that the transition probabilities do not depend on time t, while the Markov chain requirement ensures that these probabilities do not depend on the history of the process.

section, we consider θ_b as a fixed parameter) ; they both are taken care of in Section 4.

It makes sense to assume that $\theta_0 \gg \theta_b$, meaning that the discovery of a backstop would be a rather long run issue compared with the possibility of an embargo that country B is facing immediately.

2.3 The optimal stockpiling policy

In order to solve the model, let us first consider the case of a single disruption (as in Hillmann and Long [1983]), the date τ_0 of which is uncertain. Further on in the paper, we discuss the possibility of multiple embargoes interspersed by periods of trade. To deal with this permanent supply disruption, the importing country B has to design an SPR management policy that allows it to secure consumption during the embargo. The solution of the problem can be derived backwards, from the moment at which the backstop technology is made available.

2.3.1 The occurrence of the backstop technology

The discovery of the backstop makes the resource available for all $t > \tau_b$ at a constant marginal cost c. In other words, the country no more imports oil and completely frees itself from the threat of a supply disruption. At this stage, the problem becomes a static one and country B's objective at each date t is :

$$\max_{q(t)\in[0,\frac{Z}{c}]} U(q(t), Z - cq(t)) = u(q(t)) + Z - cq(t), \quad \forall \quad t > \tau_b. \tag{2.5}$$

Assuming that the country's revenue is large enough, one finds the interior solution :[8]
$q(t) = u'^{-1}(c), \quad \forall \quad t > \tau_b$.
It follows (if the backstop discovery occurs at τ_b) that the residual value of country B's

[8] $q(t) = 0$ is excluded by the assumption $u'(0) = +\infty$.

problem is given by :

$$V_b(\tau_b) = \frac{e^{-r\tau_b}}{r}[u(u'^{-1}(c)) + Z - cu'^{-1}(c)]. \tag{2.6}$$

Thus, if the backstop occurrence date was known, the representative agent's utility over time would be

$$\int_0^{\tau_b} e^{-rt} U(q(t), Z(t))dt + V_b(\tau_b), \tag{2.7}$$

which can be used to formulate the problem right after the disruption has happened.

2.3.2 Optimal depletion of the stock after the embargo

An embargo imposed on B entails that the oil consumed is drawn from the country's SPR (i.e. $m(t) = 0$ and $z(t) = Z$). Suppose the trade disruption occurs at date τ_0 with country B having stockpiled an amount $X(\tau_0)$ of oil and assume for the moment that $\tau_0 < \tau_b$. From τ_0 on, the planner's problem is to determine the optimal depletion of the stock, taking into account the fact that a backstop might be discovered at some uncertain date τ_b. In other words, he has to maximize the expected welfare. The value function associated to the problem is then :[9]

$$V_0(X(\tau_0)) = \max_{q(t)} \int_{\tau_0}^{+\infty} h_b(\tau_b - \tau_0)\{\int_{\tau_0}^{\tau_b} e^{-r(t-\tau_0)} U(q(t), Z)dt + V_b(\tau_b - \tau_0)\}d\tau_b \tag{2.8}$$

subject to : $\dot{X}(t) = -q(t)$

$q(t) \geq 0$ and $\lim_{t \to +\infty} X(t) \geq 0$

$X(\tau_0)$ predetermined.[10]

Since the problem is autonomous, the initial date of the problem can be set (without loss of generality) at $\tau_0 = 0$. Furthermore, integrating by parts, one can rewrite the ob-

[9] Using an argument similar to that of Dasgupta and Heal (1974), let us assume that the value of reserves falls to zero after the implementation of the backstop technology. This does not fundamentally affect the results, yet it simplifies the presentation.

[10] At the beginning of the embargo, the accumulated stock is known to the planner.

jective function as :

$$\int_0^{+\infty} e^{-rt} G_b(t) \left[u(q(t)) + Z\right] dt + \int_0^{+\infty} h_b(t) V_b(t) dt \,, \tag{2.9}$$

where $G_b(t) = \int_t^{+\infty} h_b(t)dt$ is the Right Tail Distribution Function (RTDF).

Notice that the first term of the sum (2.9) can be interpreted as follows : given the random process governing τ_b, the weight (in the expression of welfare) of the discounted utility enjoyed from the consumption of $q(t)$ barrels of oil is exactly the probability $G_b(t)$ that the backstop will occur at a date which is posterior to t. I shall use this interpretation again later on in the paper. As τ_b is exponentially distributed, the expression of the RTDF is $G_b(t) = e^{-\theta_b t}$.

The other terms in the sum (2.9) being deterministic, we can formulate the Hamiltonian of the problem :

$$H_0(q(t), X(t), \lambda(t), t) = e^{-(r+\theta_b)t} \, u(q(t)) - \lambda(t) q(t) \,. \tag{2.10}$$

Since $\lim\limits_{q \to 0} u'(q) = +\infty$ by assumption, we can ignore the non negativity constraint on $q(t)$ and write the necessary conditions as :

$$U_q(q(t), Z) = u'(q(t)) = e^{(r+\theta_b)t} \, \lambda(t) \tag{2.11}$$

$$\dot{\lambda}(t) = 0 \tag{2.12}$$

$$\dot{X}(t) = -q(t) \tag{2.13}$$

$$\lim_{t \to +\infty} \lambda(t) X(t) = 0 \tag{2.14}$$

By (2.12), we have : $\lambda(t) = \lambda(0) = \bar{\lambda} > 0$ at each date t.[11] Hence, the transversality

[11] The value $\bar{\lambda}$ of one barrel in stock is positive, given that the reserves allow the country to avoid zero consumption in disruption periods (recall that $\lim\limits_{q \to 0} u'(q) = +\infty$).

condition (2.14) can be rewritten as : $\lim_{t\to+\infty} X(t) = X(\tau_0) - \int_0^{+\infty} q(t)dt = 0$ (the stock should be depleted asymptotically), i.e.

$$\int_0^{+\infty} q(t)dt = X(\tau_0) . \tag{2.15}$$

The first two necessary conditions give the optimal oil consumption

$$q(t) = u'^{-1}(\bar{\lambda} e^{(r+\theta_h)t}) . \tag{2.16}$$

From the moment τ_0 at which the trade disruption occurs, the current value of one barrel from the SPR increases over time at rate $(\theta_h + r)$. This is coherent with Dasgupta and Heal (1974) : the possibility of a backstop entails an adjustment in the discount rate.

Combining equations (2.15) and (2.16) yields :

$$\int_0^{+\infty} u'^{-1}(\bar{\lambda} e^{(r+\theta_h)t})dt = X(\tau_0) . \tag{2.17}$$

Finally, recovering the value of $\bar{\lambda}$ and substituting it into equation (2.16), one gets the expression of $q(t)$. $V_0(X(\tau_0))$ can then be derived as the welfare associated to this consumption path. Since the Hamiltonian is concave in (q, X), the necessary conditions presented above are also sufficient for optimality.

Lemma 2.3.1. *$V_0(X)$ is increasing and concave.*

Proof. See Appendix.

Remark 2.3.1. *Lemma 2.3.1 shows that $V_0(X)$ exhibits the properties of a utility function : call it the value of the SPR. Since we have assumed that $\lim_{q\to 0} u'(q) = +\infty$, the same holds for the value of the SPR : $\lim_{X\to 0} V_0'(X) = +\infty$.*

The following proposition describes how the parameters affect the strategic reserves.

Proposition 2.3.1.
Suppose that $u(E) = \frac{q^{1-\sigma}}{1-\sigma}$ (with $0 < \sigma < 1$), i.e. the utility of oil exhibits constant

Elasticity of Intertemporal Substitution (EIS). Then, during the embargo :

(i) The greater the discount rate r, the lower the strategic reserves ;

(ii) The sooner country B expects the backstop to occur, the lower the reserves at any date t.

Proof. See Appendix.

Having derived the optimal oil consumption plan after the embargo has happened, one can formulate the program that the importing country has to solve from the very outset of the problem.

2.3.3 Building up the reserves

In the previous subsection, we have assumed that the occurrence of the backstop is posterior to the embargo ($\tau_0 < \tau_b$), which is not necessarily the case. Actually, the order in which the two events happen is stochastic and the social planner has to take into account the corresponding probabilities in order to set the program. Assuming that the two processes are independent, we can write :

$$prob\ [\tau_0 < \tau_b] = \int_0^{+\infty} h_0(\tau_0) G_b(\tau_0) d\tau_0 = \int_0^{+\infty} \theta_0\ e^{-(\theta_0+\theta_b)\tau_0} d\tau_0 = \frac{\theta_0}{\theta_0 + \theta_b} . \quad (2.18)$$

Let us denote :

$$\pi = prob\ [\tau_0 < \tau_b] = \frac{\theta_0}{\theta_0 + \theta_b} , \quad (2.19)$$

$$\bar{\pi} = prob\ [\tau_b \leq \tau_0] = \frac{\theta_b}{\theta_0 + \theta_b} . \quad (2.20)$$

If the embargo happens first, welfare can be expressed as :

$$\int_0^{+\infty} e^{-rt} [G_0(t) U(q(t), Z(t)) + h_0(t) V_0(X(t))] dt , \quad (2.21)$$

where $V_0(X(t))$ is the value function derived from the previous subsection. Notice that this expression obtains by using the same method as for equation (2.9) : the current discounted utility is weighted by the probability $G_0(t)$ that the embargo occurs after t.

Similarly, if the backstop occurs first,[12] then country B ceases its oil imports and the embargo threat becomes irrelevant. The utility enjoyed over the whole horizon is then given by :

$$\int_0^{+\infty} [e^{-rt}G_b(t)U(q(t),Z(t)) + h_b(t)V_b(t)]dt, \tag{2.22}$$

where $V_b(t)$ is the utility (derived earlier) enjoyed after the discovery, at date t, of a substitute to oil.[13]

Taking the expected value and replacing the RTDFs by their expressions, the planner's problem is therefore :

$$V_1 = \max_{q(t),\, z(t)} \int_0^{+\infty} e^{-rt}\{(\pi e^{-\theta_0 t} + \bar{\pi} e^{-\theta_b t})U(q(t),z(t)) + \pi\theta_0 e^{-\theta_0 t}\; V_0(X(t))\}dt \tag{2.23}$$

$$\begin{aligned} \text{subject to :}\quad & \dot{X}(t) = -q(t) + \tfrac{Z-z(t)}{p(t)} \\ & X(t) \geq 0,\; q(t) \geq 0,\; 0 \leq z(t) \leq Z \\ & X(0) = 0 \;\text{ and }\; \{p(t),\, t \geq 0\} \;\text{ given.} \end{aligned}$$

Notice that, on the optimal path, the only constraint that might be binding at some date t is $0 \leq z(t)$. Indeed, due to the assumption that $\lim_{q\to 0} u'(q) = +\infty$, the other constraints are always satisfied with strict inequalities. The Hamiltonian of the problem can be written as :

$$\begin{aligned} H(q(t),X(t),\mu(t),t) = e^{-rt}\{(\pi e^{-\theta_0 t} + \bar{\pi} e^{-\theta_b t})U(q(t),z(t)) + \pi\theta_0 e^{-\theta_0 t}\; V_0(X(t))\} \\ + \mu(t)\left(-q(t) + \frac{Z-z(t)}{p(t)}\right) \end{aligned} \tag{2.24}$$

[12]Even if the order of occurrence is stochastic, the case $(\tau_b \leq \tau_0)$ is less likely to happen due to our assumption $\theta_0 \gg \theta_b$.

[13]Recall that V_b is expressed in discounted value.

Recalling that $U_z(q(t),z(t)) = 1$, necessary conditions are given by :

$$e^{-rt}(\pi e^{-\theta_0 t} + \bar{\pi} e^{-\theta_b t})u'(q^*(t)) = \mu^*(t) \tag{2.25}$$

$$\mu^*(t) = e^{-rt}(\pi e^{-\theta_0 t} + \bar{\pi} e^{-\theta_b t})p(t) \tag{2.26}$$

$$-\dot{\mu}^*(t) \leq \pi\theta_0 e^{-(r+\theta_0)t}\, V_0'(X^*(t)) \text{ (with equality if } z^*(t) > 0) \tag{2.27}$$

$$\dot{X}^*(t) = -q^*(t) + \frac{Z - z^*(t)}{p(t)} \tag{2.28}$$

Since U and V_0 are concave functions of (q,z,X), the Hamiltonian is concave as well. This guarantees that the necessary conditions are sufficient to determine the optimal stockpiling policy. Equations (2.25) and (2.26) summarize the instantaneous trade-off between energy consumption, oil stockpiling and the composite commodity consumption. In particular, this requires that the marginal utility $u'(q(t))$ of oil consumption be always equal to the price $p(t)$ along the optimal path,[14] and hence determines oil consumption, $q^*(t)$.

Condition (2.27), with the equality, is the no-arbitrage condition guaranteeing that country B cannot benefit from increasing (or decreasing) the reserves. Indeed, increasing by one unit the level of the stock at date t would result in a (discounted) gain of utility $e^{-rt}V_0'(X^*(t))$ if an embargo were to happen the moment after (recall that this gain must be weighted by the probability $\pi\theta_0 e^{-\theta_0 t}$ that the embargo occurs at this date, prior to the backstop). Hence, optimality requires this expected gain $\pi\theta_0 e^{-(r+\theta_0)t}V_0'(X^*(t))$ to be exactly offset by the change (a loss in this case) $\dot{\mu}^*(t)$ in the marginal value of the reserves. Given that V_0 is known and that $\dot{\mu}^*(t)$ obtains from equation (2.26), the optimal value X^* of the stock can be derived from (2.27) at each date. The following lemma and proposition describe the optimal SPR management.

[14]Notice that the condition holds only for importing countries endowed with a high enough revenue. Indeed, for some very poor countries, the entirety of the budget would be spent on current energy consumption and the optimal path would rather be characterized by : $u'(q(t)) \geq p(t)$ (with the strict inequality at some dates) and $Z - p(t)q(t) = 0$ (for these dates).

Lemma 2.3.2. *On the optimal path, suppose that (2.27) is satisfied with equality at a given moment t_0. Then from date t_0 on, the SPR, X^*, optimally decreases through time. Furthermore, at any date $t > t_0$, (2.27) will also hold with equality.*

Proof. Replacing $p(t)$ by its value and taking the derivative of (2.26) with respect to time yields :

$$\dot{\mu}^*(t) = -p(0)(\theta_0 \pi e^{-\theta_0 t} + \theta_b \bar{\pi} e^{-\theta_b t}).$$

This, substituted into inequality (2.27), requires that the optimal path satisfy :

$$V_0'(X^*(t)) \geq \underbrace{p(0)e^{rt}\left[1 + \frac{\theta_b^2}{\theta_0^2} e^{(\theta_0 - \theta_b)t}\right]}_{f(t)} \qquad \left(\text{because } \frac{\bar{\pi}}{\pi} = \frac{\theta_b}{\theta_0}\right). \tag{2.29}$$

Since (2.27) is satisfied with equality at date t_0, we have :

$$-\dot{\mu}^*(t_0) = \pi \theta_0 e^{-(r+\theta_0)t_0} \, V_0'(X^*(t_0)).$$

Next, by way of contradiction, suppose that we have $X^*(t) \geq X^*(t_0)$ for some $t > t_0$. Given that V_0' is decreasing and $f(t)$ is increasing,[15] we necessarily have

$$V_0'(X^*(t)) \leq V_0'(X^*(t_0)) = f(t_0) < f(t).$$

This violates condition (2.29). Thus, $\dot{X}^*(t) < 0$.

In addition, $-\dot{\mu}^*(t) < \pi \theta_0 e^{-(r+\theta_0)t} \, V_0'(X^*(t))$ means that it is desirable for the country to raise the SPR (since the marginal utility of the stock is greater than the decrease $-\dot{\mu}^*(t)$ of the value of marginal oil consumption). Hence, decreasing the reserves is not optimal and we necessarily have $\dot{X}^*(t) \geq 0$. Thus, assuming that (2.27) is satisfied with strict inequality yields a contradiction with what we have proved in the previous paragraph. It follows that (2.27) is satisfied with equality at date t.

[15]To see why $f(t)$ is increasing, recall that $\theta_0 > \theta_b$.

Proposition 2.3.2. *There exists a date $T \geq 0$ before which the country keeps on increasing the reserves (as long as the disruption has not happened). From T on, the level of the SPR decreases asymptotically over time towards 0.*

Proof. Recall that X^* represents the optimal SPR path prior to the embargo. Suppose first that condition (2.27) holds with equality from date $t_0 = 0$ on, i.e.

$$V_0'(X^*(0)) = p(0)\frac{\pi\theta_0 + \bar{\pi}\theta_b}{\pi\theta_0} = p(0)\left(1 + \frac{\theta_b^2}{\theta_0^2}\right). \tag{2.30}$$

Then, from Lemma 2.3.2, we know that $\dot{X}^*(t) < 0$ for any $t > 0$. Thus, it suffices to take $T = 0$.[16]

On the other hand, let (2.27) be satisfied with strict inequality at date 0, i.e.

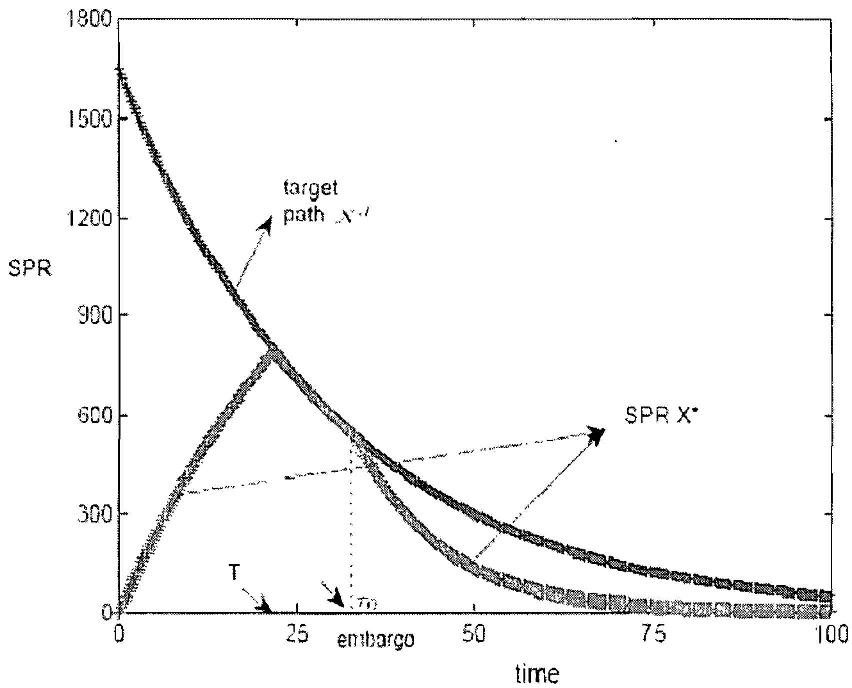
$$V_0'(X^*(0)) > p(0)\left(1 + \frac{\theta_b^2}{\theta_0^2}\right).$$

Then, it is optimal to increase the stock, since the cost of acquiring one unit of SPR is less than the value of the reserves if an embargo happens in the next instant. Therefore, we necessarily have $\dot{X}^*(0) > 0$. Denote by X^d the path where (2.29) is satisfied with equality at each date t. $X^d(t)$ is defined by $V_0'(X^d(t)) = f(t)$. Since f increases to infinity, it follows from Lemma 2.3.1 and Remark 2.3.1, that $X^d(t)$ decreases to 0. Given that $X^*(0) < X^d(0)$ and $\dot{X}^* > 0$ (as long as (2.27) holds with strict inequality), continuity requires that there exists $T > 0$ such that $X^*(T) = X^d(T)$ (see Figure 2.1). From date T on, the two curves coincide (by Lemma 2.3.2) and decrease to 0. ■

Remark 2.3.2. *Notice that T is actually a function of the budget Z. Indeed, during the building up stage ($\dot{X}^*(0) > 0$), we know that $z^*(t) = 0$; the budget used to increase the*

[16] In this first case, the solution is interior. Since the reserves $X^*(0)$ are purchased at date 0, it follows from (2.30) and the budget constraint that : $Z \geq p(0)[V_0'^{-1}\{p(0)(1 + \frac{\theta_b^2}{\theta_0^2})\} + u'^{-1}(p(0))]$. If this condition is not satisfied, then the solution is not always interior : at early dates, country B would like to hold higher reserves that it can possibly purchase.

FIG. 2.1 – The SPR pattern $X(t)$



reserves is then given by $Z - p(t)u'^{-1}(p(t))$. Thus, T is defined by :

$$\int_0^T \left[\frac{Z}{p(t)} - u'^{-1}(p(t))\right] dt = X^d(T) . \tag{2.31}$$

It follows that $T = \hat{T}(Z)$ is a decreasing function of Z, the importing country's budget. The greater the monetary endowment of the country, the shorter the building up of the SPR.

After date T, the reserves decrease ; and the decline becomes even steeper if an embargo occurs (see Figure 2.1), since current oil consumption is directly drawn from the SPR.

These results differ qualitatively from those of Bergström, Loury and Persson (1985) who find a constant level of SPR. The reason for this difference is that the price is increa–

sing at the rate of interest to satisfy the Hotelling rule, instead of remaining constant. Indeed, with a rising price, the trade-off between oil and the composite good turns in favor of the latter as time goes by (because oil becomes relatively more and more expensive). It follows that country B's oil consumption decreases to zero, as well as the SPR needed to secure this consumption. Figure 2.1 (obtained from simulations) illustrates the SPR pattern described in the previous proposition.

Having derived $X^*(t)$, one can retrieve $z^*(t)$ from the budget constraint (2.28). Finally, the value function V_1 is obtained as the welfare associated to this optimal path. For the class of constant EIS utility functions, the following result shows how a change in the parameter θ_0 affects the reserves X^*.

Proposition 2.3.3. *Prior to trade disruption, we have the following :*
(i) $\frac{\partial X^*(t)}{\partial \theta_0} \geq 0$, *for all dates t and all θ_0 such that $t \leq \frac{1}{\theta_0}$;*
(ii) $\frac{\partial X^*(t)}{\partial \theta_0} < 0$, *for all dates t and all θ_0 such that $t > \frac{1}{\theta_0}$.*

Proof. See Appendix.

Following an infinitesimal increase in θ_0, country B optimally raises the SPR for all dates that are smaller than $\frac{1}{\theta_0}$, the average waiting time for an embargo to occur. On the contrary, for all dates that are greater than the average waiting time, the country optimally reduces the SPR.

2.3.4 The case of multiple trade disruptions

Up to now, we have assumed that the importing country is anticipating a single trade disruption which lasts forever. Let us now consider how country B will design its SPR management policy if it expects several embargoes interspersed with free trade intervals.

The durations of the regimes (either embargo or free trade) follow the processes described by (2.3). The number of embargoes is a random variable which takes the values $k = 1, 2, .., n, ...$ The problem can be solved by induction. Assuming that the optimal SPR management with $n-1$ disruptions is known (i.e. V_{n-1} is given), the planner has to solve the two following problems :[17]

$$W_{n-1}(X)=\max_{q(t)}\int_0^{+\infty}e^{-rt}\{(\pi_1 e^{-\theta_1 t}+\bar{\pi}_1 e^{-\theta_b t})U(q(t),Z)+\pi_1\theta_1 e^{-\theta_1 t}\,V_{n-1}(X(t),p(t))\}dt \tag{2.32}$$

subject to : $\dot{X}(t)=-q(t)$

$X(t)\geq 0,\ q(t)\geq 0$

$X(0)=X$ given

and

$$V_n(X)=\max_{q(t),z(t)}\int_0^{+\infty}e^{-rt}\{(\pi e^{-\theta_0 t}+\bar{\pi}e^{-\theta_b t})U(q(t),z(t))+\pi\theta_0 e^{-\theta_0 t}\,W_{n-1}(X(t))\}dt \tag{2.33}$$

subject to : $\dot{X}(t)=-q(t)+\frac{Z-z(t)}{p(t)}$

$X(t)\geq 0,\ q(t)\geq 0,\ z(t)\leq Z$

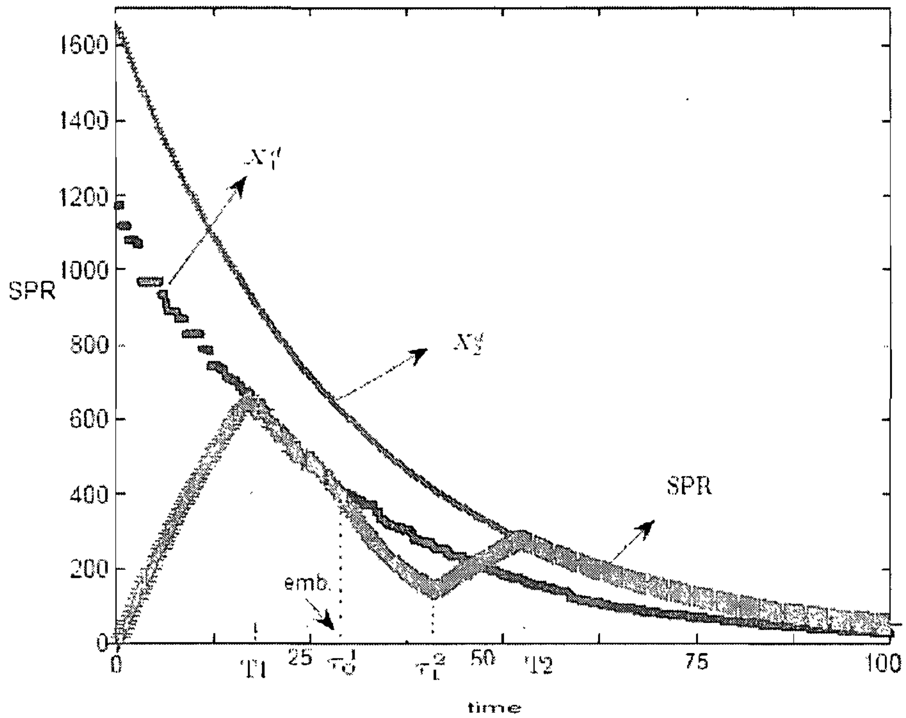
$X(0)=X$ given and $\{p(t),\ t\geq 0\}$ given.

Problem (2.32) represents the maximization of country B's welfare during the first embargo (i.e. the number of embargoes that are still to come after trade resumes is $n-1$). Hence, the continuation value is V_{n-1}, which is known. Problem (2.33) represents the problem that the planner has to solve in order to derive the optimal management of the SPR before the first embargo. The continuation value (if the first embargo occurs) is W_{n-1}, which is obtained from the solution of (2.32).

Knowing $W_0=V_0$ and V_1 (that we have obtained in subsections 3.2 and 3.3), we can derive W_1 and V_2, and then W_2 and V_3... Thus, using induction, the optimal management of the SPR can be determined for any anticipated number n of disruptions. However, it becomes harder to retrieve the analytical expressions of both the value function and the optimal SPR. Figure 2.2 (obtained from simulations) shows what would be the optimal stockpiling policy with $n=2$.

[17] Similar to π and $\bar{\pi}$, π_1 and $\bar{\pi}_1$ are given by : $\pi_1=prob[\tau_0^1<\tau_b]=\frac{\theta_1}{\theta_1+\theta_b}$ and $\bar{\pi}_1=prob[\tau_0^1\geq\tau_b]=\frac{\theta_b}{\theta_1+\theta_b}$.

FIG. 2.2 – Multiple disruptions



Similar to the case of a single disruption, let us denote by X_1^d (resp. X_2^d) the path where the decrease in the shadow value of the reserves is exactly compensated by the marginal utility of the stock prior to the first embargo (resp. prior to the second embargo). From date $t = 0$, country B will keep on accumulating precautionary reserves until the date T_1 at which the SPR curve hits the curve X_1^d. From then on, the level of the stock optimally decreases with time. The decline is even sharper during the embargo (which occurs at τ_0^1), due to the fact that current consumption is drawn directly from the SPR. From date τ_2^1 at which trade resumes after the first embargo, the country starts building up the reserves (in anticipation of the second embargo) until date T_2, at which the actual SPR and the target path X_2^d coincide. From T_2 on, the reserves optimally decrease. And so forth (if the country expects further disruptions).

2.4 Strategic investment in R& D

In the previous sections, we have assumed that the probability θ_b of the backstop occurring at date t is given exogenously. Recalling the model where a single embargo is expected, let us consider a framework in which the country has the possibility of engaging in *R&D* in order to influence this parameter. Then, the probability of the substitute occurring within the interval $[t, t+dt]$ depends on country B's effort, y; and the moment at which the backstop will be discovered follows an exponential distribution of parameter $\theta_b(y)$, with $\theta_b^{'}(y) > 0$. The instantaneous cost of effort is $C(y)$. To keep things simple, we will assume $\theta_b(y) \equiv y$ and $C(y) \equiv y^2$. I will also assume that the level of effort remains constant within any given regime (embargo or free trade), thus maintaining the stationarity of each process. Recall that the state s of the market takes two values : $s = 0$ during an embargo and $s = 1$ during a free trade period. Let us denote by y_0 (resp. y_1) the country's effort in research during the embargo (resp. during the free trade regime).

During the regime $s \in \{0, 1\}$, country B chooses the effort y_s (and hence θ_b) which maximizes its expected welfare :

$$\max_{y_s \geq 0} \quad V_s(y_s, X(\tau_s)) - \int_0^{+\infty} e^{-(r+y_s)t}\, C(y_s)dt, \tag{2.34}$$

where V_s $(s = 0, 1)$ are the value functions V_0 and V_1 determined in Section 3 (see (2.8) and (2.23)) and $X(\tau_s)$ is the level of the reserves at the beginning of the regime s.

Proposition 2.4.1. *It is always optimal for the importing country to undertake R&D : the country will choose a positive effort, no matter what the state of the market (i.e. $y_s^* > 0$).*

Proof. The overall cost of choosing the level of effort y_s (which is the second term in the objective function (2.34)) can be rewritten as :

$$\int_0^{+\infty} e^{-(r+y_s)t}\, C(y_s)\, dt = \frac{C(y_s)}{r+y_s} = \frac{y_s^2}{r+y_s}$$

The first order condition (for an interior solution) is then given by :

$$\frac{\partial V_s(y_s, X(\tau_s))}{\partial y_s} - \frac{y_s^2 + 2ry_s}{(r+y_s)^2} = 0 \tag{2.35}$$

Condition (2.35) equates the marginal benefit of effort with its marginal cost.

I prove in the Appendix that $\frac{\partial V_s(y_s=0)}{\partial y_s} > 0$ and $\lim_{y_s \to +\infty} \frac{\partial V_s(y_s)}{\partial y_s} = 0$. Thus, at $y_s = 0$, the marginal benefit of effort exceeds its marginal cost (which is zero). Therefore, it is optimal for the country to increase the effort in research (there is no corner solution). For $y_s \to +\infty$, the marginal cost of effort is greater than its marginal benefit. Thus, continuity requires that there exists a $y_s^* > 0$ such that (2.35) holds and the second order for a maximum is satisfied. It is therefore optimal for the importing country to engage in *R&D* by choosing the positive effort y_s^*. ■

During the regime s, the effort optimal effort y_s^* that the importing country optimally invests in *R&D* depends on the size $X(\tau_s)$ of the precautionary stock at the beginning of the regime. The following result describes this relation.

Proposition 2.4.2. *During the regime s, the optimal effort y_s^* is a decreasing function of the SPR, $X(\tau_s)$: the less the reserves at the beginning of the regime, the greater the effort invested in research.*

Proof. Recall the first order condition (2.35) that determines y_s^*. Since $\frac{\partial^2 V_s(y_s^*, X(\tau_s))}{\partial y_s \partial X(\tau_s)} < 0$ (see Appendix) and $\frac{\partial^2 V_s(y_s^*, X(\tau_s))}{\partial y_s^2} - \frac{2r^2}{(r+y_s^*)^3} < 0$ (by the second order condition), taking the total derivative of this condition yields :

$$\frac{dy_s^*}{dX(\tau_s)} = -\left(\frac{\partial^2 V_s(y_s^*, X(\tau_s))}{\partial y_s \partial X(\tau_s)}\right) / \left(\frac{\partial^2 V_s(y_s^*, X(\tau_s))}{\partial y_s^2} - \frac{2r^2}{(r+y_s^*)^3}\right) < 0. \quad ■$$

Hence, the effort in research decreases with the level of the SPR held at the beginning of the regime. Notice that the greatest investment in research corresponds to the case where the SPR stock is zero at the beginning of the regime. This might happen for instance if the country has no stock at the beginning of the problem ($X(0) = 0$) and the embargo occurs right at the outset ($\tau_0 = 0$).

2.5 Concluding remarks

This paper presents a simple framework to determine the optimal stockpiling policy for a country which imports an essential nonrenewable resource and is threatened with supply disruptions. We show the existence of a "target path" for the SPR. The building up stage, then, consists in increasing the precautionary stock as fast as allowed by the limited budget, until the SPR reaches the desired target. Once this first stage is over, the country will stay on this decreasing path unless a disruption occurs. We also derive the optimal depletion of the SPR while an embargo is going on ; it depends on the country's rate of impatience, on the effort invested in research and also on the expected duration of the disruption.

The paper points out the necessity for the importing country to invest in *R&D* in order to ensure, eventually, the permanent availability of the substitute. This incentive to develop a backstop increases with the decline of the SPR.

We acknowledge, however, that some issues have not been explicitly dealt with in this paper. First, our model solves for the optimal stockpiling from the social planner's point of view : we have abstracted from issues such as private storage. Some have studied the question of public vs private storage. Williams and Wright (1982) argue that public storage (or regulation) might discourage firms from holding stocks. Second, our model does not deal with the randomness in price (as we assume its perfect predictability). As discussed in Hamilton (2008) and Wu and McCallum (2005), it is not an easy task to predict oil prices with accuracy, even in the short run. By dealing with a deterministic price pattern (representing the trend of oil price), we have provided the reader with a benchmark scenario to understand how the optimal stockpiling policy can be derived.

Future research includes optimal SPR sharing. Indeed, more and more countries are concluding negotiations requiring each member to make its SPR available to the other parties in case of a disruption. Among other agreements, France, Italy and Germany have accepted to share their reserves stocks in case of an emergency. Japan, South Korea, and New Zealand (to a certain extent) have also reached such an agreement in 2007. How do these (binding ?) agreements compare to individual stockpiling policies ?

CHAPITRE 3

THE ECONOMICS OF OIL, BIOFUEL AND FOOD COMMODITIES

3.1 Introduction

The recent food crisis has become a major concern for world leaders. In June 2008, the World Food Summit organized by the United Nations that took place in Rome raised many questions about the causes of this crisis. Indeed, from the beginning of this decade until late 2008, major food crop prices have increased for the first time since the 1970s. The prices of corn, rice, wheat as well as other crops reached record highs. According to a recent article by the Economist magazine,[1] food accounts in Botswana and South Africa for a fifth of the consumer price index ; in Sri Lanka and Bangladesh it accounts for two-thirds. This might explain the recent violent clashes that took place in several developing countries (Haiti, Cameroon and Egypt, among others) in the wake of the sharp increase in crop prices that occurred in 2007 and 2008.

Against this backdrop, a number of explanations for this crisis have been proposed. First, a line of argument attributes the increase in major crop prices to the rising world demand for food, which has not been followed by adequate investments in the agricultural sector. The proponents of this view, namely the UN secretary general, declared that global food output must increase by 50% by 2030 in order to maintain 'food security'. However, such an argument suffers from a drawback. While the lack of investments in agriculture has been a long-term structural problem ever since the end of the 'first green revolution' of the 1960s and 70s, it is the case that the recent rise in crop prices has been sharp and dramatic. An alternative view considers that the recent development of the biofuel industry has a lot to do with the food crisis. Advocates of this view include a number of specialized NGOs and renowned international research organizations, like the

[1] From The Economist print edition, June 5, 2008, page 70.

International Food Policy Research Institute (IFPRI). According to the IFPRI, biofuels account for up to 30% of the increase in the price of agricultural commodities.

From 1999 until the summer of 2008, both global energy demand[2] and fossil fuels prices have been steadily rising. This has caused pressure for the development of biofuels[3] as an alternative source of energy. This was not the case during the 1990s, when the fossil fuel price was too low to allow for the economic viability of this renewable resource. This increase in the demand for biofuels has generated a 'crowding-out effect' in the agricultural sector. Many argue that scarce agricultural resources are being diverted away from food production towards the production of biofuels, which results in a reduction in global crop supplies.

The fact that the prices of both oil and food commodities have tumbled as of the last quarter of 2008 also suggests that during the current decade, both prices have become highly positively correlated. In this paper we investigate, within a tractable model, the mechanisms through which these two markets are linked and how the development of the biofuel industry has affected the correlation between energy and food prices.

Since the questions arising from the introduction of biofuels are relatively recent, the economic literature on this subject is relatively limited. Moreover, as pointed out by Rajagopal and Zilberman (2007) in a World Bank policy survey, "the environmental literature is dominated by a discussion of net carbon offset and net energy gain, while indicators relating to impact on human health, soil quality, biodiversity, water depletion, etc., have received much less attention".[4] Hochman, Sexton and Zilberman (2008) study the crowding-out effect of biofuels on the agricultural sector. They propose a two-country general equilibrium trade model with energy as intermediate input. In their model, they

[2]China and India's staggering growth rates account for a large chunk of that.

[3]Not to mention environmental lobbying and political pressures that have led to an additional regulation induced demand. For instance, the government of Canada recently announced that it will impose a mandatory 5% biofuel content in each liter of gasoline sold in the local market by 2010.

[4]See Rajagopal and Zilberman (2007), page 2. They also point out that serious concerns about the carbon benefits of current biofuels can be raised, namely the fact that biofuels consume a significant amount of energy that is derived from fossil fuels. See as well Giampietro, Ulgiati and Pimentel (1997), Lal (2004), Pimentel and Patzek (2005), Farrell *et al.* (2006).

consider two sources of energy (fossil and biofuel); both the biofuel and food sectors compete for land and labor.[5] Their main results suggest that trade liberalization tends to increase the demand for energy, which decreases food production and causes losses in forests and other non-agricultural lands. They also show that neutral technical change in agricultural production, such as biotechnology and second generation biofuel technologies, mitigates this pressure on land. Chakravorty et al. (2008) deal with a related question. They find that backstop technologies will be adopted earlier than expected in response to high increases in food and petroleum prices. They also argue that, as a result, either the demand for energy will decrease or petroleum will be replaced by backstop technologies.

In the present paper, unlike Hochman et al. (2008) and following Chakravorty et al. (2008), we study the effects of nonrenewable resource exhaustion over time by a price-leading cartel as the impetus behind the rising global demand for biofuels, which might have a perverse effect on 'food security'. We consider that the finite land resource is put into two alternative uses by price-taking farmers : food and biofuel production. We abstract from the issue of global atmospheric pollution caused by emissions, unlike in the other two papers. Our main focus is on the relation between energy and food prices which follows from the depletion of fossil fuel (oil for short) and the development of biofuels as a substitute.

We present the model in Section 3.2. In Section 3.3 we solve the farmers' land allocation problem. Section 3.4 is devoted to the oil cartel's optimal depletion and pricing decisions. We conclude in Section 3.5.

[5] In this paper, for simplicity, we consider that only the land resource is shared between food and energy productions. As a matter of fact, many resources are subject to trade-off between these two sectors. See for instance Gaudet, Moreaux and Withagen (2006), where water is shared between oil and agriculture.

3.2 The model

Consider an economy composed of an agricultural sector and an oil sector and of two markets, one for energy and one for food. The energy market is supplied by farmers, in the form of biofuel, and by an oil cartel. The market for food is supplied by price-taking farmers.

3.2.1 The supply sides

The cartel, acting as a dominant firm, extracts fossil fuel and sells it on the energy market. The finite stock of nonrenewable fossil fuel at date t is $S(t)$. We assume that the stock is homogeneous. We also assume that the size of the stock is known with certainty and we abstract from energy storage issues. This stock is depleted at the rate $E_f(t)$, at zero cost, for simplicity. The evolution of the stock is given by :

$$\dot{S}(t) = -E_f(t) \quad \forall\, t. \tag{3.1}$$

The total amount of productive land available is also finite. We assume a representative farmer whose behavior summarizes the production decisions of the mass of all farmers. This representative farmer owns a parcel of arable land of size L. He has to decide how to allocate his land between the production of food and the production of biofuel. The food production of the representative farmer is denoted $Q(t)$ while the amount of biofuel he produces is denoted $E_b(t)$, measured in oil equivalent. At each date t, the fixed amount of arable land is allocated between food and biofuel, so that :

$$L_a(t) + L_b(t) = L, \tag{3.2}$$

where $L_a(t)$ and $L_b(t)$ stand for the amounts of land allocated respectively to food and biofuel.

We will assume that one unit of oil, or its equivalent in the form of biofuel, generates one unit of energy.[6] Therefore, the total supply of energy, $E(t)$, will be :

$$E(t) = E_f(t) + E_b(t). \tag{3.3}$$

Food output is given by

$$Q(L_a(t)) = AL_a(t), \tag{3.4}$$

while biofuel output is given by

$$E_b(L_b(t)) = BL_b(t). \tag{3.5}$$

The constants A and B are conversion parameters related to the technology in use. Parameter B reflects the (linear) conversion efficiency into biofuel of the biomass produced using one unit of land.[7] We assume that the farmers incur increasing marginal cost of production. Specifically, producing $Q(L_a(t))$ and $E_b(L_b(t))$ will cost respectively $\frac{c_a}{2}(AL_a(t))^2$ and $\frac{c_b}{2}(BL_b(t))^2$, where c_a and c_b are positive cost parameters.

3.2.2 The demand sides

The demand for energy at date t is given by the following :

$$E(t) = N(t)(\bar{p}_e - p_e(t)), \tag{3.6}$$

where $N(t)$ is the population at date t and $p_e(t)$ is the price of energy. The inverse demand is thus given by

$$p_e(t) = \bar{p}_e - \frac{E(t)}{N(t)}. \tag{3.7}$$

[6]Of course it should be understood that the biofuel production represents here the net energetic equivalent of the biomass produced by the farmers. Indeed, in order to produce biofuel, fossil energy is required at various stages (see Rajagopal and Zilberman (2007), p. 34).

[7]For example, in the case of sugarcane one hectare of land yields 4900 liters of ethanol (see Rajagopal and Zilberman (2007), p. 102).

We assume that population grows at a constant rate $\gamma \geq 0$ i.e. $N(t) = N_0 e^{\gamma t}$. The world demand for food at date t is given by :

$$Q(t) = N(t)\left(\bar{p}_a - p_a(t)\right), \tag{3.8}$$

where $p_a(t)$ is the price of food.

The parameters $\bar{p}_e$ and $\bar{p}_a$ represent the choke prices in the energy and food markets respectively.

3.3 The farmers' problem

In this section, we solve the land allocation problem faced by the representative farmer. In the energy market, farmers act as a competitive fringe vis-a-vis the oil cartel. The energy price $p_e(t)$ is set by the cartel and this price is taken as given by the representative farmer. The representative farmer also takes as given the price $p_a(t)$ in the food market.

The representative farmer maximizes the sum of his food and biofuel profits subject to the land constraint (3.2). In other words, at any date t :

$$\max_{L_a(t),\, L_b(t)} \left[p_e(t)BL_b(t) + p_a(t)AL_a(t) - \frac{c_b}{2}(BL_b(t))^2 - \frac{c_a}{2}(AL_a(t))^2\right]$$

subject to $L_a(t) + L_b(t) = L$.

Replacing L_b by $L - L_a$, the first-order condition for the determination of L_a can be written, assuming an interior solution :

$$p_a(t)A - c_a A^2 L_a(t) = p_e(t)B - c_b B^2[L - L_a(t)]. \tag{3.9}$$

It says that the allocation of land to food production must be such that it equalizes the marginal net benefit from allocating land to either of its two usages. From (3.9) we get the solution for land allocation to food production as a function of the the two prices :

$$L_a(p_a(t), p_e(t)) = \frac{p_a(t)A - p_e(t)B + c_b B^2 L}{c_a A^2 + c_b B^2}. \tag{3.10}$$

Therefore, recalling (3.4), food supply is given by

$$Q^S(p_a(t), p_e(t)) = AL_a(p_a(t), p_e(t)). \tag{3.11}$$

It then follows from (3.2) that

$$L_b(p_a(t), p_e(t)) = \frac{p_e(t)B - p_a(t)A + c_a A^2 L}{c_a A^2 + c_b B^2}, \tag{3.12}$$

and, from (3.5), biofuel supply is :

$$E_b^S(p_a(t), p_e(t)) = BL_b(p_a(t), p_e(t)). \tag{3.13}$$

We will assume that

$$c_b B^2 L > \bar{p}_e B - \bar{p}_a A > -c_a A^2 L. \tag{3.14}$$

This guarantees that we have an interior solution, so that positive quantities of land will be allocated to both food and biofuel. Indeed, the full marginal cost of land allocation to biofuel, given that it can also be used for food production, is $c_b B^2 L_b(t) + p_a(t)A - c_a A^2 [L - L_b(t)]$. When neither food nor biofuel is produced ($L_b = L_a = 0$), this reduces to $\bar{p}_a A - c_a A^2 L$ and assumption (3.14) guarantees that there exists a positive $L_b(t)$ which equates the full marginal cost to $p_b(t)B$, the marginal revenue from land allocation to biofuel. Similarly, the full marginal cost of land allocation to agriculture is $c_a A^2 L_a(t) + p_b(t)B - c_b B^2 [L - L_a(t)]$ and, by the same reasoning, assumption (3.14) guarantees that the solution for $L_a(t)$ is interior.

Given the energy price $p_e(t)$ set by the cartel, the market clearing condition, obtained by equating the demand for food (given by (3.8)) with the supply of food (given by (3.11)), yields the equilibrium food price as a function of the energy price :

$$p_a(p_e(t)) = \frac{\bar{p_a}(c_aA^2 + c_bB^2)N(t) - c_bAB^2L + ABp_e(t)}{A^2 + (c_aA^2 + c_bB^2)N(t)}. \tag{3.15}$$

Thus, because of the competition for the limited amount of land between the production of food and of biofuel, the price of food is linked to the price of energy. As can be seen from (3.15), at any date t, the higher the price of energy, the higher the price of food.

Using (3.15), the biofuel supply at any date t can now be rewritten as a function of $p_e(t)$ only :

$$E_b^S(p_e(t)) = BL_b(p_a(p_e(t)), p_e(t)). \tag{3.16}$$

3.4 The oil cartel

Subtracting the farmers supply of biofuel (3.16) from the total energy demand (3.6) gives the residual demand faced by the oil cartel :

$$E_f(p_e(t)) = N(t)\left(\bar{p_e} - p_e(t)\right) - BL_b(p_a(p_e(t)), p_e(t)) \tag{3.17}$$

Applying equations (3.12) and (3.15) in (3.17), one can derive the inverse residual demand which can be written as :

$$p_e\left(E_f(t)\right) = \beta(t) - \alpha(t)\frac{E_f(t)}{N(t)}, \tag{3.18}$$

where

$$\beta(t) = \frac{\theta\bar{p}_eN(t)^2 + A(A\bar{p}_e + B\bar{p}_a - ABLc_a)N(t) - A^2BL}{\theta(N(t))^2 + (A^2 + B^2)N(t)} < \bar{p}_e \tag{3.19}$$

$$\alpha(t) = \frac{A^2 + \theta N(t)}{A^2 + B^2 + \theta N(t)} > 0, \tag{3.20}$$

and $\theta = A^2c_a + B^2c_b > 0$.

Observe that $\beta(t)$ can be viewed as a time-varying effective choke price for the residual demand facing the cartel at each date t. Because of the presence of a fringe of biofuel producers, $\beta(t)$ is smaller than $\bar{p_e}$, the choke-price of total demand for energy. The cartel has to set a price that is lower than $\beta(t)$ if it wants to sell positive amounts of oil. When $p_e(t) \geq \beta(t)$, the total demand for energy is met exclusively by the biofuel producers. As for $\alpha(t)$, it gives the time-variant slope of the residual inverse demand for oil faced by the cartel.

It can be directly established from (3.20) that $\alpha(t)$ is increasing over time if population is growing and that $\lim\limits_{t\to+\infty} \alpha(t) = 1$. As for $\beta(t)$, its time derivative is given by $\dot{\beta}(t) = (\partial\beta/\partial N)\dot{N}(t)$, where

$$\frac{\partial\beta}{\partial N} = \frac{B\left[(A^4 + A^2B^2 + 2\theta A^2 N)L + (B\bar{p_e} - A\bar{p_a} + c_aA^2L)\theta N^2\right]}{(N(t))^2\left[A^2 + B^2 + \theta N(t)\right]^2}. \tag{3.21}$$

The right-hand side of (3.21) is positive, since, from (3.14), $B\bar{p}_e - A\bar{p_a} + c_aA^2L > 0$. Therefore $\beta(t)$ also increases over time as long as population is growing and $\lim\limits_{t\to+\infty} \beta(t) = \bar{p_e}$. Note that the residual demand for energy converges asymptotically to the total demand for energy.

We will assume that $N_0 > \tilde{N}$, where $\tilde{N}$ is the positive root of $\beta(0) = 0$, so that $\beta(t) > 0$ for all $t \geq 0$. Since by assumption the marginal cost of oil production is zero, this guarantees that oil production will be positive from the outset.

Given the inverse residual demand, the oil cartel chooses its oil production path and the date of exhaustion of its oil stock so as to maximize its discounted flow of profits :

$$\max_{E_f(t),\, T} \int_0^T e^{-rt}\left(\beta(t) - \alpha(t)\frac{E_f(t)}{N(t)}\right)E_f(t)\; dt$$

subject to :

$$\dot{S}(t) = -E_f(t),$$

$$E_f(t) \geq 0,$$

$$S(0) = S_0 \text{ and } S(T) \geq 0\,.$$

The Hamiltonian of the problem is :

$$H\left(E_f(t), \lambda(t), t\right) = e^{-rt}\left(\beta(t) - \alpha(t)\frac{E_f(t)}{N(t)}\right)E_f(t) \; - \lambda(t)E_f(t)$$

and the following conditions are necessary for optimality :

$$\beta(t) - 2\alpha(t)\frac{E_f(t)}{N(t)} - e^{rt}\lambda(t) \leq 0, \quad \left(\beta(t) - 2\alpha(t)\frac{E_f(t)}{N(t)} - e^{rt}\lambda(t)\right)E_f(t) = 0 \quad (3.22)$$

$$\dot{\lambda}(t) = 0 \quad (3.23)$$

$$\dot{S}(t) = -E_f(t) \quad (3.24)$$

$$\left(\beta(T) - \alpha(T)\frac{E_f(T)}{N(T)} - e^{rT}\lambda(T)\right)E_f(T) = 0 \quad (3.25)$$

$$\lambda(T) \geq 0 \;\text{ and }\; \lambda(T)S(T) = 0. \quad (3.26)$$

The Hamiltonian being concave in the control variable $E_f(t)$, linear in $\lambda(t)$ and independent of the state variable $S(t)$, conditions (3.22) to (3.26) are also sufficient for optimality.

Condition (3.22) says that, if at any date t extraction is positive, the profit derived from the marginal barrel of oil must be equal to its current *in situ* value, $e^{rt}\lambda(t)$.

From (3.23), we have that $\lambda(t) = \lambda(0) = \bar{\lambda}$ for all $t \in [0, T]$. The current shadow value of in situ oil therefore grows at the rate of interest, so that no profitable arbitrage is possible with respect to the stock of oil.

The transversality condition (3.25) states that the value of marginally delaying the terminal date T, which is given by the Hamiltonian evaluated at T, must be zero. Notice that for any values of $E_f(T) \neq 0$, conditions (3.22) and (3.25) cannot both hold at the terminal date T. Therefore the optimal rate of extraction at T must be zero : $E_f(T) = 0$.

The transversality condition (3.26) states that the value of the remaining stock at the terminal date T must be zero, either because $\lambda(T) = \bar{\lambda} = 0$, or $S(T) = 0$, or both. But $\bar{\lambda} = 0$ would, from (3.22), contradict the fact that $\beta(T) > 0$. It follows that $\bar{\lambda} > 0$ and $S(T) = 0$: the oil stock will be exhausted. Since the choke price is finite, this will occur in finite time.

Recalling that $N(t) = e^{\gamma t} N_0$, exhaustion of the stock means that :

$$\int_0^T \frac{\beta(t) - \bar{\lambda} e^{rt}}{2\alpha(t)} e^{\gamma t} dt = \frac{S_0}{N_0}. \tag{3.27}$$

This, along with

$$\bar{\lambda} = e^{-rT} \beta(T), \tag{3.28}$$

uniquely determines $\bar{\lambda}$ and T as functions of the per-capita initial oil stock, S_0/N_0. For instance, in the case where population is constant ($\gamma = 0$ and $N(t) = N_0$), substituting for $\bar{\lambda}$ from (3.28) into (3.27), we find that T is given by :

$$rT + e^{-rT} = \frac{2\alpha r}{\beta} \frac{S_0}{N_0} + 1. \tag{3.29}$$

The solution for $\bar{\lambda}$ then follows from (3.28).

The cartel's oil extraction path is therefore given by :

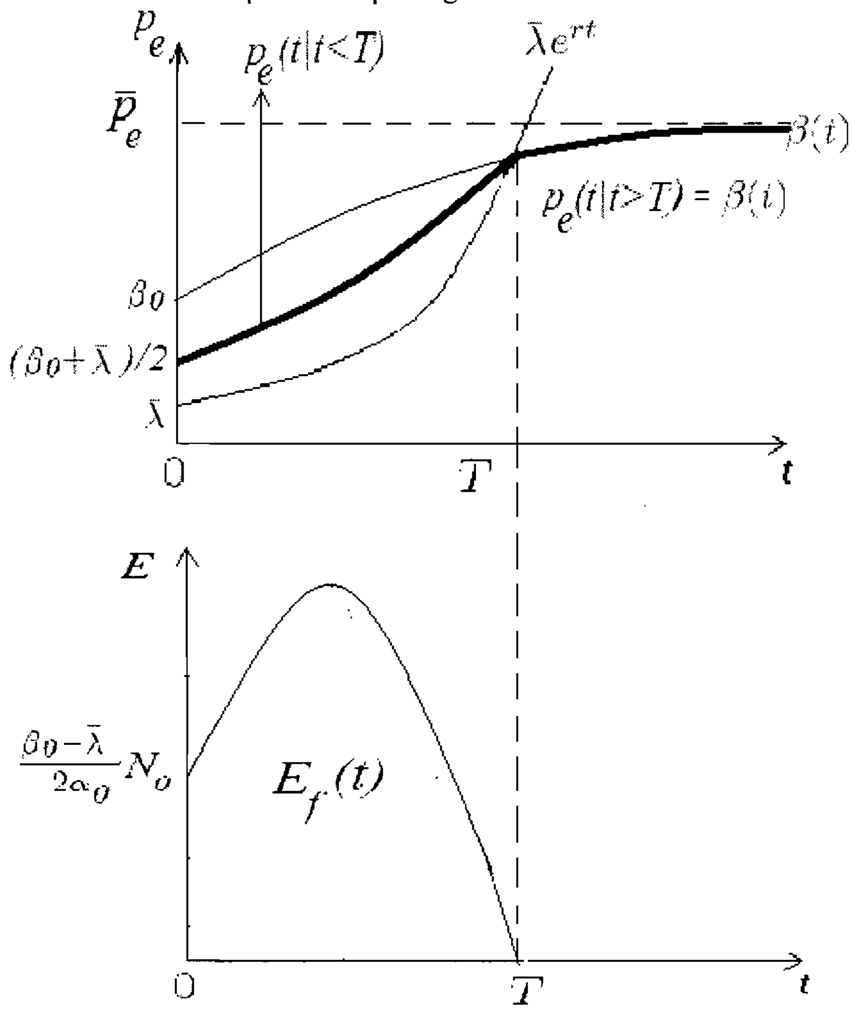
$$E_f(t) = \frac{\beta(t) - \bar{\lambda} e^{rt}}{2\alpha(t)} N(t) \qquad \forall t \in [0, T], \tag{3.30}$$

where $\bar{\lambda}$ and T are the solutions for the shadow price of oil and the date of exhaustion of the stock in terms of S_0/N_0. Hence, recalling (3.18), the evolution of the price of energy over time will be given by :

$$p_e(t) = \beta(t) - \alpha(t)\frac{E_f(t)}{N(t)} = \begin{cases} \dfrac{\beta(t) + \bar{\lambda}e^{rt}}{2} & \forall\, t \in [0, T] \\ \beta(t) & \forall\, t \in [T, \infty). \end{cases} \qquad (3.31)$$

Since both $\beta(t)$ and $e^{rt}\lambda$ are increasing functions of time, the price of energy rises

FIG. 3.1 – Optimal oil pricing and extraction of the cartel



continuously over time. At date T, $e^{rT}\bar{\lambda} = \beta(T)$, so that $p_e(T) = \beta(T) < \bar{p_e}$ and the stock of oil is exhausted. From date T on, energy demand is supplied exclusively by the biofuel producers, with its market price equal to $\beta(t)$ and tending asymptotically to $\bar{p}_e$, as long as the population is growing.

If the population is constant, then so is $\beta(t)$ and so will be the price of energy for all $t > T$. In all cases however, because the presence of the biofuel fringe lowers the price leader's effective choke price, the oil cartel will choose to exhaust its stock before price reaches $\bar{p}_e$ and hence will exhaust its stock of oil sooner than it would in the absence of the fringe. The switch at T from energy being supplied from both oil and biofuel to biofuel only results in a downward jump in the rate of change of the price of energy at T and hence a kink in its time path. This is illustrated in the top graph of Figure 3.1.

As for the rate of oil extraction by the cartel, although it must eventually be decreasing to reach zero at T, it cannot be ruled out that it be increasing at the beginning, as illustrated in the bottom graph of Figure 3.1. Differentiating (3.31) with respect to time, we find that :

$$\dot{E}_f(t) = \frac{(\dot{\beta}(t) - re^{rt}\bar{\lambda})N(t) + (\beta(t) - e^{rt}\bar{\lambda})\dot{N}(t)}{2\alpha(t)} - \frac{(\beta(t) - e^{rt}\bar{\lambda})N(t)\dot{\alpha}(t)}{2\alpha(t)^2}. \qquad (3.32)$$

Since the second term is positive, for $E_f(t)$ to be increasing the first term must also be positive. Therefore, in order for oil production to be increasing over some initial interval of time, it is necessary, though not sufficient, that :

$$\left(\dot{\beta}(0) - r\bar{\lambda}\right)N_0 + \left(\beta(0) - \bar{\lambda}\right)\gamma N_0 > 0. \qquad (3.33)$$

In the particular case of a constant population ($\gamma = 0$), we have $\dot{\beta}(t) = 0$ for all t and the necessary condition (3.33) cannot be satisfied. Therefore, if the population is constant, the production of energy from fossil fuel will be at its maximum at $t = 0$ and will decrease from thereon until it reaches zero at $t = T$. By continuity, the same will be true for some small values of γ.

As for the price path of food, substituting for $p_e(t)$ from (3.31) into (3.15), it can be written :

$$p_a(t) = \begin{cases} \dfrac{\theta \bar{p}_a N(t) + (AB/2)(\beta(t) + \bar{\lambda} e^{rt}) - AB^2 Lc_b}{A^2 + \theta N(t)} & \forall t \in [0, T] \\[2ex] \dfrac{\theta \bar{p}_a N(t) + (AB/2)\beta(t) - AB^2 Lc_b}{A^2 + \theta N(t)} & \forall t \in [T, \infty). \end{cases}$$

Differentiating with respect to time, its evolution over time can be written :

$$\dot{p}_a(t) = \begin{cases} \left[\dfrac{\partial p_a}{\partial N} + \dfrac{1}{2}\dfrac{\partial p_a}{\partial p_e}\dfrac{\partial \beta}{\partial N}\right] \dot{N}(t) + \dfrac{1}{2} r \bar{\lambda} e^{rt} \dfrac{\partial p_a}{\partial p_e} & \forall t \in [0, T] \\[2ex] \left[\dfrac{\partial p_a}{\partial N} + \dfrac{1}{2}\dfrac{\partial p_a}{\partial p_e}\dfrac{\partial \beta}{\partial N}\right] \dot{N}(t) & \forall t \in [T, \infty). \end{cases} \tag{3.34}$$

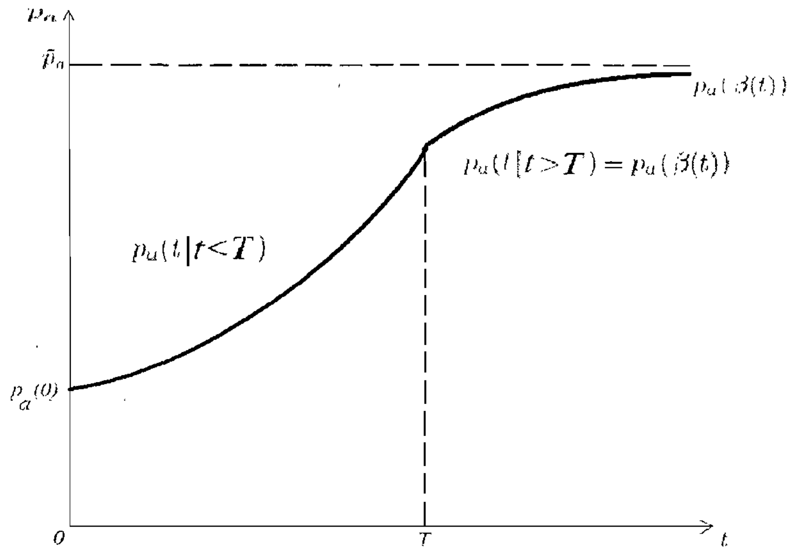
As already pointed out, $\partial p_a / \partial p_e$ is positive, from (3.15). Therefore the second term in the top expression is positive. Also, as established from (3.21), $\partial \beta / \partial N$ is positive. As for $\partial p_a / \partial N$, it is given by :

$$\begin{aligned} \frac{\partial p_a}{\partial N} &= \frac{\theta[(\theta\{N(t) - 1\})\bar{p}_a + A^2 \bar{p}_a - ABp_e(t) + c_b AB^2 L]}{[A^2 + \theta N(t)]^2} \\ &\geq \frac{\theta A[A\bar{p}_a - B\bar{p}_e + c_b B^2 L]}{[A^2 + \theta N(t)]^2} > 0 \quad \text{(by assumption (3.14))}. \end{aligned}$$

Therefore the price of food is continuously increasing if the population is growing, as illustrated in Figure 2. It will grow at a faster rate for $t < T$, while the oil stock is being depleted, than for $t > T$, when the only source of supply of energy is biofuel, with a kink in the path occurring at T. If population is constant, it will grow until T and become constant afterwards.

For all $t > T$, the farmers will be the sole suppliers of both the food and the energy market. The equilibrium prices can then be determined using the solution to the farmers' land allocation problem of Section 3.3 and the market clearing conditions. We will have $p_a(t) = p_a(p_e(t))$ given by (3.15), but with $p_e(t) = \beta(t)$. The price of food will tend

FIG. 3.2 – The food price path



asymptotically to $\bar{p}_a$, while the price of energy tends asymptotically to $\bar{p}_e$. In the case of a constant population, both of those prices would be constant beyond T, both smaller than their respective choke price.

3.5 Concluding remarks

The object of this paper has been the effect on the food sector of the recent development of biofuels as a substitute to oil in the supply of energy. We have shown how the competition for the finite land resource, which takes place between biofuel and food production, explicitly defines a relationship between the energy price and the food price. The rate of depletion of the oil stock may at first increase if population is growing, but it will eventually decrease to zero as the stock gets exhausted. The price of energy will however increase continuously while the stock of oil is being depleted, due to decline of the remaining *per capita* stock of oil and hence whether population is growing or constant. If population is growing, it will keep increasing after biofuel becomes the only source of energy.

As for the food price, it is also increasing. Two effects account for this growth in the price of food. Firstly, the increase in energy price raises the opportunity cost of the use of land for food production, creating an incentive for farmers to reallocate their land in favor of biofuel production. Secondly, population growth increases the demand for food, thus pushing upwards the equilibrium price in the food market.

The model provides an insight into the so-called "food security" issue. It would seem that investment in productivity enhancing measures in the agricultural sector could, by increasing crops, contribute to dampening the rise of the food price. In addition, measures aimed at reducing the demand for energy would directly reduce the upward pressure on the price of energy and indirectly that on the price of food. Subsidies to food production might also be required in order to maintain a minimum viable level of food production with a growing population. Controlling population would also be a means of reducing the pressure in both the energy and the food markets.

CONCLUSION GÉNÉRALE

Les différents sujets traités dans cette thèse illustrent bien les approches normative et positive qui caractérisent la science économique. Le premier essai de la thèse s'est penché sur une question axiomatique, à savoir la détermination des mécanismes de partage qui doivent être utilisés si l'on veut satisfaire l'exigence d'équité que constitue l'axiome "Ranking". L'intérêt de ce sujet réside dans le fait qu'il existe de nombreux problèmes de partage pour lesquels l'ensemble des agents est constitué de sous-groupes comportant des individus similaires : tarification des services d'électricité, de téléphone, d'internet (classification par secteur géographique), partage des coûts du système de sécurité sociale (classification par niveau de revenu), affectation des quotas d'émissions à des pays ou à des régions (classification par niveau d'industrialisation),... Pour ces problèmes, il nous semble crucial, d'un point de vue normatif, de pouvoir comparer les parts payées par les agents qui influencent le coût du projet de façon identique (c'est-à-dire par les agents qui appartiennent à un même sous-groupe). Les résultats obtenus dans le premier chapitre de la thèse permettent de sélectionner les mécanismes répondant à cette nécessité. Les deux autres chapitres de la thèse participent essentiellement de l'approche positive. Ils expliquent les mécanismes de décision des agents, notamment du pays menacé d'embargo (deuxième chapitre) et des agriculteurs produisant du biocarburant (troisième chapitre). Dans ces deux essais, nous décrivons les équilibres dynamiques qui découlent des comportements de maximisation des agents.

De façon plus spécifique, le premier chapitre a établi que pour le cas de deux agents, une méthode de partage de coûts vérifie l'axiome "Ranking" si et seulement si le flot qui la représente est symétrique dans le carré défini à partir de la plus petite des demandes. Nous avons montré qu'avec trois agents et plus, la symétrie du flot est également nécessaire mais n'est plus suffisante pour que la méthode satisfasse "Ranking". Pour finir, nous avons montré que dans la classe des flots fixes élémentaires, l'axiome "Ranking" est caractérisé par la propriété suivante : dans chaque tranche, si le flot commence à se rapprocher de la diagonale, alors il doit continuer sa progression vers cette diagonale, ceci avec au moins la même intensité. Des exemples détaillés et des figures nous ont permis d'examiner le comportement des méthodes de partage les plus connues vis-à-vis de l'axiome "Ranking" : la méthode Aumann-Shapley ne satisfait pas cet axiome, tandis que la méthode sérielle et celle de Shapley-Shubik le vérifient toutes les deux.

Dans le deuxième chapitre, nous déterminons la politique optimale de gestion des réserves stratégiques pour un pays qui importe une ressource non renouvelable (telle que le pétrole) et qui est susceptible de subir des embargos dont les dates de début et de fin sont aléatoires. Nous montrons que la constitution des réserves staratégiques est effectuée sur une longue période, plutôt que de façon instantanée. Nous établissons l'existence d'une courbe décroissante qui représente le niveau "idéal" de réserves que le pays importateur aimerait maintenir au fil du temps. Seulement, étant donné que les réserves stratégiques sont inexistantes au départ, le pays n'est clairement pas sur cette courbe à la date initiale. Sa meilleure stratégie est donc d'accroître les réserves aussi rapidement que possible (compte tenu du budget disponible), ceci jusqu'à ce que la courbe représentant le niveau réel des réserves coïncide avec le "sentier idéal". A partir de ce moment, la phase de constitution des réserves stratégiques est achevée et le pays doit, de façon optimale, diminuer le niveau de ses réserves (tout en restant sur ce "'sentier idéal"). Nous avons également montré qu'il est de l'intérêt du pays importateur d'investir dans la recherche d'une source d'énergie renouvelable qui, lorsqu'elle sera opérationnelle, viendra remplacer les importations de pétrole, affranchissant ainsi le pays de la menace d'embargo. Le niveau optimal d'investissement dans la recherche a également été caractérisé.

Finalement, le troisième chapitre a analysé l'effet du développement des biocarburants sur les prix des produits alimentaires. Cet essai propose un cadre d'analyse dynamique intégrant le marché de l'énergie et celui des biens alimentaires. Il explique comment, compte tenu des prix, les agriculteurs répartissent la terre arable entre les productions de biens alimentaires et de biocarburants. La relation liant le prix des aliments à celui de l'énergie a été formellement mise en évidence. Cette relation montre qu'à un instant donné, un cours du pétrole plus élevé se traduit par des produits alimentaires plus chers. L'analyse a également montré qu'une forte croissance de la population pourrait inciter le cartel à extraire des quantités croissantes de pétrole dans un premier temps. Toutefois, le pic d'extraction est ensuite atteint et la production de pétrole retombe pour finalement atteindre zéro à la date terminale d'extraction, qui survient en temps fini. L'examen de l'évolution des prix (sur les marchés de l'énergie et des aliments) montre que ceux-ci augmentent continuellement pendant la phase d'extraction du pétrole. Dans le cas où la population est constante, les prix se stabilisent après l'épuisement du pétrole. Si par contre la population est croissante, alors les deux prix continuent de croître après la date terminale d'extraction.

ANNEXE A

APPENDIX TO CHAPTER 1

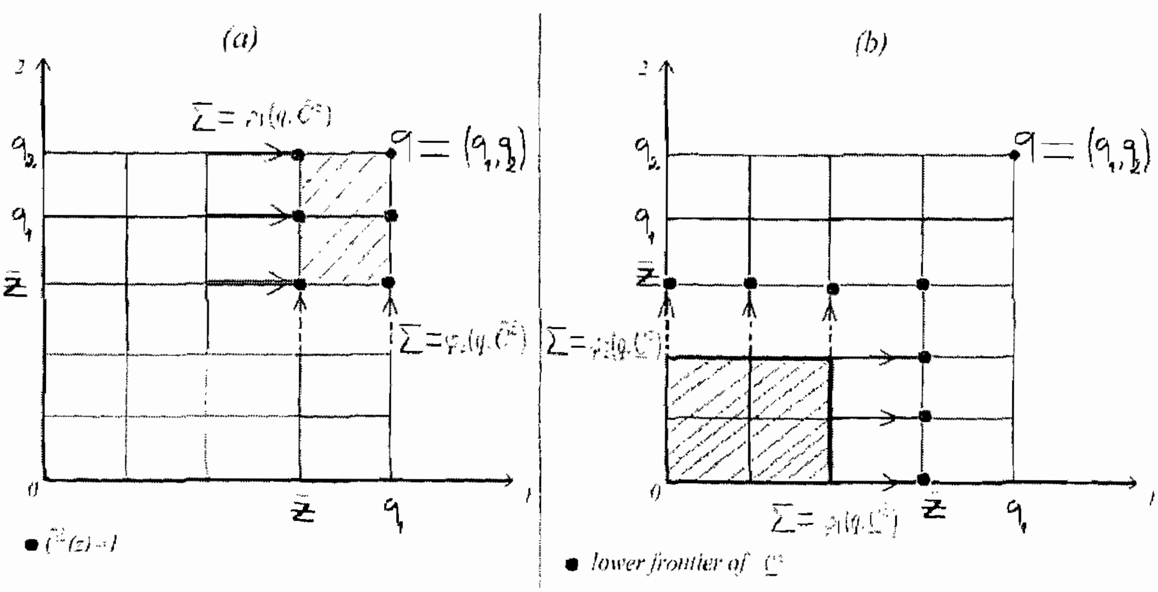
A.1 Proof of Lemma 1.3.1

Throughout the following proofs, we use cost functions taking the values 0 or 1 only. By convention, we represent every such function C by describing the set $X^C = \{z \in \mathbb{N}^2 / C(z) = 1\}$, or sometimes its lower frontier. Let us fix $q \in [(0,0);(\overline{q},\overline{q})]$, with $q_1 \leq q_2$. For any $\overline{z}$ such that $0 < \overline{z} \leq q_1$, consider the symmetric cost functions $\widetilde{C^{\bar{z}}}$ and $\underline{C}^{\bar{z}}$ defined by :

$$\widetilde{C^{\bar{z}}}(z) = \begin{cases} 1 & if\ z \geq (\overline{z},\overline{z}) \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \underline{C}^{\bar{z}}(z) = \begin{cases} 0 & if\ z \leq (\overline{z},\overline{z}) \\ 1 & \text{otherwise.} \end{cases}$$

See Figure A.1 for illustration. By Ranking (respectively on $\widetilde{C^{\bar{z}}}$ and $\underline{C}^{\bar{z}}$), we have :

FIG. A.1 – $\widetilde{C^{\bar{z}}}$ and $\underline{C}^{\bar{z}}$



$$\varphi_1(q,\overline{C^{\bar{z}}}) = \sum_{z_2=\bar{z}}^{q_2} f_1(z_1,z_2) \leq \frac{1}{2} \leq \sum_{z_1=\bar{z}}^{q_1} f_2(z_1,z_2) = \varphi_2(q,\overline{C^{\bar{z}}}) \tag{A.1}$$

$$\varphi_1(q,\underline{C}^{\bar{z}}) = \sum_{z_2=0}^{\bar{z}-1} f_1(z_1,z_2) \leq \frac{1}{2} \leq \sum_{z_1=0}^{\bar{z}-1} f_2(z_1,z_2) = \varphi_2(q,\underline{C}^{\bar{z}}) \tag{A.2}$$

Since $\varphi_1(q,\overline{C^{\bar{z}}})+\varphi_1(q,\underline{C}^{\bar{z}}) = \sum_{z_2=0}^{q_2} f_1(z_1,z_2) = 1 = \sum_{z_1=0}^{q_1} f_2(z_1,z_2) = \varphi_2(q,\overline{C^{\bar{z}}})+\varphi_2(q,\underline{C}^{\bar{z}})$, the result (1.6) of the lemma follows. ■

A.2 Proof of Proposition 1.3.1 ($\Rightarrow$) : with $n=2$, the symmetry of the flow is necessary for Ranking.

By Lemma 1.3.1, we have $f_1(1,0) = f_2(0,1) = \frac{1}{2}$. Next, proceed by induction. Suppose $f_1(z_1,z_2) = f_2(z_2,z_1)$ for all $z \in [(0,0),(q_1,q_1-1)]$ such that $z_1+z_2 \leq k-1$ (induction hypothesis *H1*) and show that the result still holds for $z \in [(0,0),(q_1,q_1-1)]$ such that $z_1+z_2 \leq k$.

Case 1 : k is even ($k=2p$)

By induction hypothesis, the bold flow (see in Figure A.2-(a)) is symmetric. In the following, we will very often omit the components of the two cost shares that are equal by induction hypothesis.

$$\longrightarrow \text{Show first that } f_1(p,p) = f_2(p,p) \tag{A.3}$$

- $f_1(p,p) > f_2(p,p) \Rightarrow f_1(p+1,p-1) > f_2(p-1,p+1)$ by flow conservation and induction hypothesis H1, contradicting Ranking on the symmetric cost function C^1 (see Figure A.2-(a)).
- $f_1(p,p) < f_2(p,p) \Rightarrow f_1(p+1,p-1) < f_2(p-1,p+1)$ (flow conservation)

$$\Rightarrow \sum_{t=0}^{p-1} f_1(p+1,t) < \sum_{t=0}^{p-1} f_2(t,p+1)$$

$$\Rightarrow f_1(p+1,p) > f_2(p,p+1), \tag{A.4}$$

since Lemma 1.3.1 implies that $\sum_{t=0}^{p} f_1(p+1,t) = \frac{1}{2} = \sum_{t=0}^{p} f_2(t,p+1)$.
Furthermore, we have :

$$f_1(p,p) < f_2(p,p) \;\Rightarrow\; \sum_{t=p+1}^{q_2} f_1(p,t) > \sum_{t=p+1}^{q_1} f_2(t,p). \tag{A.5}$$

Combining (A.4) and (A.5) contradicts Ranking on C^2, which proves (A.3). By flow conservation, we get from (A.3) that $f_1(p+1,p-1) = f_2(p-1,p+1)$.

$\rightarrow$ Show next that $f_1(p+s,p-s) = f_2(p-s,p+s)$ for $0 < s \leq min\{p,q_1-p\}$ (A.6)

From what we have previously shown, we know that (A.6) holds for the case $s = 1$. We use a second induction argument. Suppose that $f_1(p+j,p-j) = f_2(p-j,p+j)$ by induction hypothesis $H2$. This means that the bold flow in the diagram representing C^3 in Figure A.2-(c) is symmetric. It suffices to prove that $f_1(p+j+1,p-j-1) = f_2(p-j-1,p+j+1)$ for $0 < j < j+1 \leq min\{p,q_1-p\}$.

- $f_1(p+j+1,p-j-1) > f_2(p-j-1,p+j+1) \Rightarrow f_1(p+j,p-j) > f_2(p-j,p+j)$, which yields a contradiction on C^3 (see Figure A.2-(c), where $j = 1$).
- $f_1(p+j+1,p-j-1) < f_2(p-j-1,p+j+1) \Rightarrow f_1(p-j,p+j) < f_2(p+j,p-j)$

$$\Rightarrow \sum_{t=p+j+1}^{q_2} f_1(p-j,t) > \sum_{t=p+j+1}^{q_1} f_2(t,p-j) \text{ by flow conservation.} \tag{A.7}$$

Also,

$$f_1(p+j+1,p-j-1) < f_2(p-j-1,p+j+1) \Rightarrow \tag{A.8}$$
$$\sum_{t=p-j}^{p+j} f_1(p+j+1,t) > \sum_{t=p-j}^{p+j} f_2(t,p+j+1),$$

since applying Lemma 1.3.1 with $\bar{z} = p+j+1$ implies that
$\sum_{t=0}^{p+j} f_1(p+j+1,t) = \frac{1}{2} = \sum_{t=0}^{p+j} f_2(t,p+j+1)$.
Combining (A.7) and (A.8) contradicts Ranking on C^4 (see Figure A.2-(d)), which proves the desired equality, and hence (A.6). Furthermore, induction hypothesis $H1$, the flow conservation and (A.6) imply that $f_1(p-j,p+j) = f_2(p+j,p-j)$. ■

Case 2 : k is odd ($k = 2p+1$).

By induction hypothesis $H1$, the flow under the line $\sum_{i=1,2} z_i$ (see Figure A.2-(e)) is symmetric.

$$\rightarrow \text{Show first that } f_1(p+1,p) = f_2(p,p+1) \tag{A.9}$$

This follows directly from Lemma (1.3.1) and the induction hypothesis $H1$:
$f_1(p+1,p) = \frac{1}{2} - \sum_{t=1}^{p} f_1(p+1,t) = \frac{1}{2} - \sum_{t=1}^{p} f_2(t,p+1) = f_2(p,p+1)$

$\rightarrow$ Show next that $f_1(p+s+1,p-s) = f_2(p-s,p+s+1),\ 0 \leq s \leq min\{p,q_1-p-1\}$

To prove this equality, one can proceed exactly as we did for (A.6), notably by recalling the induction hypothesis $H2$ and applying Ranking to the cost functions C^5 and C^6 described in Figure A.2. ■

A.3 Proof of Proposition 1.3.1 ($\Leftarrow$) : with $n = 2$, the symmetry of the flow is sufficient for Ranking.

The proof of the *sufficiency* part of Proposition 1.3.1 unfolds in three steps, the first one introducing some new concepts and notations.

A.3.1 Preliminaries

Let X be a set such that $\emptyset \neq X \subseteq [(0,0),q]$, $q \in [(0,0);(\bar{q},..\bar{q})]$. We define :

- $\underline{Fr}(X) = \{z \in X / \exists\, i \in \{1,2\}$ such that $z - e_i \in [(0,0),q] \setminus X\}$
- $\overline{Fr}(X) = \{z \in X / \exists\, i \in \{1,2\}$ such that $z + e_i \in [(0,0),q] \setminus X\}$
- $f_-(X) = \begin{cases} \sum\limits_{z \in \underline{Fr}(X)} \sum\limits_{i \in \{1,2\} \text{ s.t. } z - e_i \in [0,q] \setminus X} f_i(z) & if\ (0,0) \notin X \\ 1 \quad \text{otherwise} \end{cases}$
- $f_+(X) = \begin{cases} \sum\limits_{z \in \overline{Fr}(X)} \sum\limits_{i \in \{1,2\} \text{ s.t. } z + e_i \in [(0,0),q] \setminus X} f_i(z + e_i) & if\ q \notin X \\ 1 \quad \text{otherwise} \end{cases}$

$f_-(X)$ and $f_+(X)$ represent, respectively, the flow entering the set X and the flow exiting the set X (see Figure A.3).

The two following results can be shown :

$$\forall\, X \text{ such that } \emptyset \neq X \subseteq [(0,0),q], \text{ we have : } f_-(X) = f_+(X). \tag{A.10}$$

$$\forall\, X_1, X_2 \text{ such that : } \emptyset \neq X_1, X_2 \subseteq [(0,0),q], \text{ we have : } X_1 \subseteq X_2 \Rightarrow f(X_1) \subseteq f(X_2), \tag{A.11}$$

where $f(X) = f_-(X) = f_+(X)$.

The first result states that, for each subset, the incoming flow is equal to the outgoing flow. Thus, we can define $f(X) = f_-(X) = f_+(X)$. As for statement (A.11), it claims that the measure of the flow crossing a subset X is an increasing function of the size of

X. We do not provide a detailed proof of the proposition, it follows from the definition of a flow. Let us describe the steps that one can follow to obtain the desired results.

- Notice that (A.10) is true for any subset X such that $|X| = 1$, by flow conservation (equation (1.4)).
- Show next that the result (A.10) extends to all subsets X such that $|X| = k+1$ (assuming that it is true for subsets with cardinality less than or equal to k.)

One can prove (A.11) in a similar way by using induction over $|X_2 \setminus X_1|$.

A.3.2 Proving Ranking for the 0-1 cost functions

Let f, the flow to $q = (q_1, q_2)$, be such that condition (1.7) is satisfied, i.e. :

$$\forall\ (z_1,z_2) \in [(0,0);(q_1,q_1-1)],\ f_1(z_1,z_2) = f_2(z_2,z_1)$$

Then, for any symmetric cost function C taking the values 0 or 1 only, we have : $\varphi_1(C) \leq \varphi_2(C)$.

Indeed, let us consider a symmetric cost function C defined from $[(0,0);(\bar{q},..\bar{q})]$ to $\{0,1\}$.

- If $C(q) = 0$, then $\varphi_1(C) = 0 = \varphi_2(C)$.
- If $C(q) = 1$, then $X^C \neq \varnothing$. Recalling the definition of a flow method, one can write : $\forall\ i = 1,2:$

$$\begin{aligned}\varphi_i(q,C) &= \sum_{z\in[e_i,q]} f_i(z)[C(z)-C(z-e_i)] \\ &= \sum_{\substack{z\in[e_i,q] \\ C(z)=1,\ C(z-e_i)=0}} f_i(z)[C(z)-C(z-e_i)] \ \text{(since } C(z)\in\{0,1\}) \\ &= \sum_{\substack{z\in \underline{Fr}(C) \\ z-e_i\in[(0,0),q]\setminus X^C}} f_i(z)[C(z)-C(z-e_i)] \\ &= \sum_{\substack{z\in \underline{Fr}(C) \\ z-e_i\in[(0,0),q]\setminus X^C}} f_i(z)\end{aligned}$$

From now on, we consider three cases that are mutually exclusive and encompass all possibilities.

1. $\underline{C(0,q_1)=1}$ See Figure A.4-(a) (recall that $q_1 \leq q_2$).

Combining the symmetry and the monotonicity of the cost function C yields $\{z \in [(0,0),q]/\ (z_1 \geq q_1)\ or\ (z_2 \geq q_1)\} \subseteq X^C$. Hence, all the variations of the cost function occur inside the square $[(0,0),(q_1,q_1)]$ and $\{z \in X^C/\ z-e_1 \in [(0,0),q] \setminus X^C\} = \{z \in \underline{Fr}(C)/\ z-e_1 \in [(0,0),q] \setminus X^C\}\} \subseteq [(0,0),(q_1,q_1-1)]$.

It follows then that :

$$\begin{aligned}\varphi_1(C) &= \sum_{\substack{z \in \underline{Fr}(C) \\ z-e_1 \in [(0,0),q] \setminus X^C}} f_1(z) \\ &= \sum_{\substack{z \in \underline{Fr}(C) \\ z-e_2 \in [(0,0),q] \setminus X^C}} f_2(z) \quad \text{[combining the symmetry of } C \text{ with (1.11)]} \\ &= \varphi_2(C)\ .\end{aligned}$$

In this case, the desired property is satisfied with equality.

2. $\underline{C(0,q_1) = 0 \text{ and } C(q_1,q_1) = 1}$ See Figure A.4-(a)

   Then, $\exists\ k \in 1,..,q_1$ such that $C(k,q_1) = 1$ and $C(k-1,q_1) = 0$. Notice that since C is symmetric, we also have : $C(q_1,k) = 1$ and $C(q_1,k-1) = 0$. It follows that :

$$\varphi_1(C) = \sum_{\substack{z \in [(0,0),(q_1,q_1-1)] \cap X^C \\ z-e_1 \in [(0,0),q] \setminus X^C}} f_1(z) + \sum_{\substack{z \in [(0,q_1);(q_1,q_2)] \cap X^C \\ z-e_1 \in [(0,0),q] \setminus X^C}} f_1(z)$$

$$\varphi_2(C,x) = f_2(x_1,k) + \sum_{\substack{z \in [(0,0);(q_1-1,q_1)] \cap X^C \\ z-e_2 \in [(0,0),q] \setminus X^C}} f_2(z) + \sum_{\substack{z \in [(0,q_1);(q_1,q_2)] \cap X^C \\ z-e_2 \in [(0,0),q] \setminus X^C}} f_2(z)$$

   From the symmetry of the flow (assumption [1.11]), one can write :

$$\sum_{\substack{z \in [(0,0);(q_1,q_1-1)] \cap X^C \\ z-e_1 \in [(0,0),q] \setminus X^C}} f_1(z) = \sum_{\substack{z \in [(0,0);(q_1-1,q_1)] \cap X^C \\ z-e_2 \in [(0,0),q] \setminus X^C}} f_2(z)$$

   Furthermore, it follows from (A.11) that :

$$\sum_{\substack{z \in [(0,q_1);(q_1,q_2)] \cap X^C \\ z-e_1 \in [(0,0),q] \setminus X^C}} f_1(z) \leq f([(0,q_1);(k-1,q_2)])$$

   (Notice that since $C(k,q_1) = 1$, monotonicity implies that $C(l,q_1) = 1$ for $k \leq l \leq q_2$. In other words $\{z \in [(0,q_1);(q_1,q_2)]/C(z) = 0\} \subseteq [(0,q_1);(k-1,q_2)]$).

   Finally, the desired result $[\varphi_1(C) \leq \varphi_2(C)]$ follows, since we have :

$$f_2(q_1,k) = \sum_{l=k,..,q_2} f_1(k,l) = f([(0,q_1);(k-1,q_2)]) \qquad \text{(see Remark 1.3.1).}$$

3. $\underline{C(q_1,q_1) = 0}$ See Figure A.4-(c) $\quad (C(q) = 1$ implies that $q_2 > q_1)$

Since $C(q) = 1$, $\exists\, k \in \{q_1+1,..,q_2\}$ such that $C(q_1,k) = 1$ and $C(q_1,k-1) = 0$ i.e. $\exists\, k \in \{q_1+1,..,q_2\}$ such that $(q_1,k) \in \underline{Fr}(C)$ and $(q_1,k-1) \in [(0,0),q] \setminus X^C$.

Hence, $$\varphi_2(C) = \sum_{\substack{z \in \underline{Fr}(C) \\ z - e_2 \in [(0,0),q] \setminus X^C}} f_2(z) \geq f_2(q_1,k)\,.$$

In order to conclude this first step, it is sufficient to prove that $f_2(q_1,k) \geq \frac{1}{2}$.

$\rightarrow$ Firstly, prove that $f_2(q_1,q_1) = \frac{1}{2}$.

Consider the segment $T_1 = [(0,q_1);(q_1,q_2)]$. Since $q = (q_1,q_2) \in T_1$, we have $f(T_1) = 1 = \sum\limits_{l=0}^{q_2} f_2(q_1,l)$.

Next, considering the square $S = [0;(q_1,q_1)]$, one can write :

$f_-(W) = f(W) = \sum\limits_{l=0}^{q_1-1} f_1(q_1,l) + \sum\limits_{l=0}^{q_1-1} f_2(l,q_2)$.

Applying (1.11) also gives $f(W) = 1 = 2\sum\limits_{l=0}^{q_1-1} f_1(q_1,l)$, thus implying that $\sum\limits_{l=0}^{q_1-1} f_1(q_1,l) = \frac{1}{2}$. Then, by flow conservation on the segment $T_2 = [0;(q_1,q_1-1)]$, we have :

$f_-(T_2) = \sum\limits_{l=0}^{q_1-1} f_1(q_1,l) = \frac{1}{2} = f_+(T_2) = f_2(q_1,q_1)$.

$\rightarrow$ Secondly, consider the segment $T_3 = [(q_1,q_1);(q_1,k-1)]$. Again, by (A.10) :

$\frac{1}{2} \leq f_-(T_3) = f_2(q_1,q_1) + \sum\limits_{l=q_1}^{k-1} f_1(q_1,l) = f_+(T_3) = f_2(q_1,k)$. ■

A.3.3 Proving Ranking for all cost functions

The last step in the proof of Proposition 1.3.1 is to generalize the previous result to the set of all (non decreasing) symmetric cost functions.

Consider $\bar{q} \in \mathbb{N}$ *such that* $q = (q_1,..,q_n) \leq (\bar{q},..,\bar{q})$. *If* C *is a nonnegative and nondecreasing function from* $[0_n,(\bar{q},..,\bar{q})]$ *to* $\mathbb{R}$ *satisfying* $C(0) = 0$, *then it can be written as :*

$$C = \sum_{i=1}^{K} \alpha_i C_i, \tag{A.12}$$

where $K \in \mathbb{N}^*$, $\alpha_i > 0$ *and* C_i *is a cost function taking the values* 0 *or* 1 *only,* $\forall\, i \in \{1,..K\}$. *Furthermore, if* C *is symmetric with respect to two coordinates, then so are all the* C_i*s.*

To see why the claim is true, one has to rank all points in $[0,(\bar{q},..,\bar{q})]$ from the lowest cost to the highest one (this is possible because they are in finite number). If $C = 0$, the result is obvious. Otherwise, let $0 < \omega_1 \leq \omega_2 \leq .. \leq \omega_K$ be the associated costs (we do not rank demands profile of which the cost is null).

→ First stage : ω_1 is the lowest positive cost. Let $P_1 = \{z \in [0_n, (\bar{q}, .., \bar{q})] / C(z) \geq \omega_1\}$. Take $C_1 = ¶_{P_1}$ (where ¶ is the indicator function) and $\alpha_1 = \omega_1 > 0$. (notice that C_1 is non decreasing, because C is ; and also, $C_1(0_n) = 0$).

→ Second stage : let $P_2 = \{z \in [0_n, (\bar{q}, .., \bar{q})] / C(z) \geq \omega_2\}$. Define $C_1 = ¶_{P_1}$ and $\alpha_1 = \omega_2 - \omega_1 > 0$.

And so on...

→ At the end of the procedure (i.e. at stage K), we have : $P_K = \{z \in [0_n, (\bar{q}, .., \bar{q})] / C(z) \geq \omega_K\}$, $C_K = ¶_{P_K}$ and $\alpha_K = \omega_K - \omega_{K-1} > 0$. And one is able to write :[1] $C = \sum_{i=1}^{K} \alpha_i C_i$.

Note that, by construction, if C is symmetric with respect to two specific coordinates, then so are all the P_is and, therefore, the C_is. ■

Now, we are set to clinch the proof of Proposition 1.3.1. Indeed, in the model with two agents, consider a symmetric cost function C as in the previous decomposition statement. The C_is are also symmetric and we can write :

$$\begin{aligned}
\varphi_1(C) &= \varphi_1\left(\sum_{i=1}^{K} \alpha_i C_i\right) \\
&= \sum_{i=1}^{K} \alpha_i \varphi_1(C_i) \\
&\leq \sum_{i=1}^{K} \alpha_i \varphi_2(C_i) \quad \text{by (A.12) and the fact that } \alpha_i > 0 \\
&= \varphi_2\left(\sum_{i=1}^{K} \alpha_i C_i\right) \\
&= \varphi_2(C)
\end{aligned}$$

which ends the proof of Proposition 1.3.1. ■

A.4 Proof of Lemma 1.3.2 and Lemma 1.3.3

Let q be such that $q_1 \leq ... \leq q_n$. The proof is derived by induction over $h = 2, .., n$.

[1] Notice that this decomposition procedure holds no matter what the number of agents.

→ The first step (with $h = 2$) actually proves Lemma 1.3.2. It can be shown, similarly to Lemma 1.3.1, by considering the two cost functions

$$\overline{C^{\bar z}}(z) = \begin{cases} 1 & if\ (z_{i1}, z_{i2}) \geq (\bar z, \bar z) \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \underline{C}^{\bar z}(z) = \begin{cases} 0 & if\ (z_{i1}, z_{i2}) \leq (\bar z, \bar z) \\ 1 & \text{otherwise.} \end{cases}$$

→ Next, we want to prove (1.10) i.e.for any direction $i \in H$ and any $\bar z$ s.t. $0 < \bar z \leq q_1$:

$$\sum_{z_{H\setminus i} \in \left[0_{H\setminus i}, (\bar z - 1)1_{H\setminus i}\right]} \ \sum_{z_{N\setminus H} \in [0_{N\setminus H}, q_{N\setminus H}]} f_i(z_i = \bar z, z_{H\setminus i}, z_{N\setminus H}) = \frac{1}{h}$$

(knowing that the property is satisfied at stage $h - 1$).

Consider the cost function defined by $C^{\bar z}(z) = \begin{cases} 0 & if\ z_i < \bar z\ \forall\, i \in H = \{i_1, i_2, .., i_h\} \\ 1 & \text{otherwise.} \end{cases}$

Denote the shares by $y_i = \varphi_i(C^{\bar z})$ (for $i = 1, .., n$) and suppose that $i_1 < .. < i_h$. By Ranking,

$$\begin{aligned} y_{i_1} &= \sum_{z_{H\setminus i_1} \in \left[0_{H\setminus i_1}, (\bar z - 1)1_{H\setminus i_1}\right]} \ \sum_{z_{N\setminus H} \in [0_{N\setminus H}, q_{N\setminus H}]} f_i(z_{i_1} = \bar z, z_{H\setminus i_1}, z_{N\setminus H}) \\ &\leq y_{i_2} \cdots \leq y_{i_{h-1}} \\ &\leq \sum_{z_{H\setminus i_h} \in \left[0_{H\setminus i_h}, (\bar z - 1)1_{H\setminus i_h}\right]} \ \sum_{z_{N\setminus H} \in [0_{N\setminus H}, q_{N\setminus H}]} f_i(z_{i_h} = \bar z, z_{H\setminus i_h}, z_{N\setminus H}) = y_{i_h}. \end{aligned} \tag{A.13}$$

See Figure A.5.

Next, consider $i_1,\ i_2 \in H$ $(i_1 < i_2)$ and the cost function

$$C'^{\bar z}(z) = \begin{cases} 1 & if\ z_i \geq \bar z\ for\ some\ i\ \in \{i_1, .., i_h\} \setminus \{i_1, i_2\} \\ 1 & if\ (z_i < \bar z\ for\ some\ i\ \in \{i_1, .., i_h\} \setminus \{i_1, i_2\})\ and\ ((z_{i1}, z_{i2}) \geq (\bar z, \bar z)) \\ 0 & \text{otherwise.} \end{cases}$$

Again, if we denote the shares by $y_i' = \varphi_i(C'^{\bar z}, q)$ (for $i = 1, .., n$), the Ranking axiom requires that :

$$\begin{aligned} y'_{i_1} &= \sum_{\substack{z_{i_2} \in [\bar{z}, q_2] \\ z_{H \setminus i_1 i_2} \in \left[0_{H \setminus i_1 i_2}, (\bar{z}-1) 1_{H \setminus i_1 i_2}\right] \\ z_{N \setminus H} \in [0_{N \setminus H}, q_{N \setminus H}]}} f_{i_1}(z_{i_1} = \bar{z}, z_{i_2}, z_{H \setminus i_1 i_2}, z_{N \setminus H}) \\ &\leq \sum_{\substack{z_{i_2} \in [\bar{z}, q_2] \\ z_{H \setminus i_1 i_2} \in \left[0_{H \setminus i_1 i_2}, (\bar{z}-1) 1_{H \setminus i_1 i_2}\right] \\ z_{N \setminus H} \in [0_{N \setminus H}, q_{N \setminus H}]}} f_{i_2}(z_{i_1}, z_{i_2} = \bar{z} z_{H \setminus i_1 i_2}, z_{N \setminus H}) = y'_{i_2} \end{aligned} \tag{A.14}$$

See Figure A.6 for illustration.

Applying the induction hypothesis on the sets $H \setminus i_2$ and $H \setminus i_2$, which are both constituted of $h-1$ elements, we get :

$$\begin{aligned} y_{i_1} + y'_{i_1} &= \sum_{\substack{z_{H \setminus i_1 i_2} \in \left[0_{H \setminus i_1 i_2}, (\bar{z}-1) 1_{H \setminus i_1 i_2}\right] \\ z_{N \setminus (H \setminus i_2)} \in [0_{N \setminus (H \setminus i_2)}, q_{N \setminus (H \setminus i_2)}]}} f_{i_1}(z_{i_1} = \bar{z}, z_{H \setminus i_1 i_2}, z_{N \setminus (H \setminus i_2)}) = \frac{1}{h-1} \\ &= \sum_{\substack{z_{H \setminus i_1 i_2} \in \left[0_{H \setminus i_1 i_2}, (\bar{z}-1) 1_{H \setminus i_1 i_2}\right] \\ z_{N \setminus (H \setminus i_1)} \in [0_{N \setminus (H \setminus i_1)}, q_{N \setminus (H \setminus i_1)}]}} f_{i_2}(z_{i_2} = \bar{z}, z_{H \setminus i_1 i_2}, z_{N \setminus (H \setminus i_1)}) = y_{i_2} + y'_{i_2}. \end{aligned}$$

From (A.13) and (A.14), it then follows that $y'_{i_1} = y'_{i_2}$ and $y_{i_1} = y_{i_2} = \frac{1}{h}$. This proves (1.10). ■

A.5 Proof of Theorem 1.3.1

Let the demand profile q satisfy $q_1 \leq \ldots \leq q_n$. We combine three induction arguments.

- The first induction is over the number of agents n. Recall that we have proved the symmetry of the flow in the model with two agents. Let us assume that (1.11) and (1.12) are true for any number of agents from 2 to $n-1$ (*induction hypothesis H0*). We want to prove that it can be extended to the case of n agents. In particular, notice that this induction hypothesis implies that for any directions $i, j, t \in N =$

$\{1,..,n\}$, we have :[2]

$$\sum_{z_t=0}^{q_t} f_i(\overline{z}_i,\overline{z}_j,z_t,\overline{z}_{-ijt}) = \sum_{z_t=0}^{q_t} f_j(\overline{z}_j,\overline{z}_i,z_t,\overline{z}_{-ijt}), \tag{A.15}$$

$\forall\ (\overline{z}_i,\overline{z}_j) \in [(0,0);(q_1,q_1-1)]$ and $\forall\ \overline{z}_{-ijt} \in [0_{n-3},q_{-ijt}]$.

- Applying equation (1.10) of Lemma 1.3.3 with $\overline{z}=1$ and $h=n$, one gets the result : $f(e_i)=\frac{1}{n}\ \forall\ i=1,..,n$. Now, suppose that our symmetry condition is met for any z such that $\sum_{i=1}^{n} z_i \leq k-1$ (*induction hypothesis H1*) and extend it to demand profiles such that $\sum_{i=1}^{n} z_i = k$
- Finally, as in the proof of the symmetry with 2 agents, we consider a third induction argument which is over the distance from the diagonal. Indeed, we first prove the symmetry for profiles which are as egalitarian as possible (*induction hypothesis H2)* and we extend the results by progressively getting away from the diagonal.

Suppose, by induction hypothesis $H1$, that we have our symmetry result for all z such that $\sum_{i=1}^{n} z_i \leq k-1$. Let us consider $k=pn+r$ $(r,p \in \mathbb{N}$ and $0 \leq r < n)$, the result of the Euclidean division of k (the sum of all demands) by n (the number of agents). Define the profile $\hat{z} = (\underbrace{p+1,..,p+1}_{r},\underbrace{p,..,p}_{n-r})$; we have $\sum_{i=1}^{n} \hat{z}_i = k$. Notice that $\hat{z}$ and all its permutations represent the most egalitarian profiles satisfying the condition that the sum of all coordinates is equal to k. Hence, if we call S_n the class of all the permutations of the set $N=\{1,..,n\}$, then the first stage of our induction argument *H2* is to prove the symmetry of the flow f for all profiles in $S(\hat{z}) = \{\sigma(\hat{z}),\ \sigma \in S_n\}$.

Define $N_p = \{i \in N\ /\ \hat{z}_i = p\}$ and $N_{p+1} = \{i \in N\ /\ \hat{z}_i = p+1\}$. Next, fix $i,j \in N$ (with $l < t$) and let σ_{ij} be the transposition relative to i and j. We want to prove that $f_j(\sigma_{ij}(\hat{z})) = f_i(\hat{z})$.

[2]Given that the symmetry of the flow is satisfied for any $n-1$ agents, this result can be shown by considering cost functions which do not vary with respect to the t-th coordinate.

Three cases describe all relevant possibilities : $\begin{cases} i \in N_{p+1} \text{ and } j \in N_p \\ i,j \in N_p \\ i,j \in N_{p+1} \end{cases}$

We discuss the first case and let the reader convince herself that the proof can be derived in a similar way for the other prospects.

Hence, suppose $i \in N_{p+1}$ and $j \in N_p$ and consider the ij-symmetric cost function defined by $C_1(z) = \begin{cases} 1 & if\ z_l > \hat{z}_l\ for\ some\ l\ \in N\setminus\{i,j\} \\ 1 & if\ \ z_{-ij} = \hat{z}_{-ij}\ and\ [(z_i \geq p+1)\ or\ (z_j \geq p+1)] \\ 0 & \text{otherwise.} \end{cases}$

The cost function C_1 is illustrated in Figure A.7. Since $q_i \leq q_j$, Ranking on C_1 requires that the shares of agents i and j satisfy the condition[3]

$\varphi_i(C_1,q) = f_i(\hat{z}) + \sum_{z_j=0}^{p} f_i(p+1,z_j,\hat{z}_{-ij}) \leq \varphi_j(C_1,q) = f_j(\sigma_{ij}(\hat{z})) + \sum_{z_i=0}^{p} f_j(z_i,p+1,\hat{z}_{-ij})$.

From $\sum_{z_j=0}^{p} f_i(p+1,z_j,\hat{z}_{-ij}) = \sum_{z_i=0}^{p} f_j(z_i,p+1,\hat{z}_{-ij})$ (which is true by induction hypothesis $H1$, see Figure A.7),[4] it follows that $f_i(\hat{z}) \leq f_j(\sigma_{ij}(\hat{z}))$.

Next, consider the cost function

$C_2^m(z) = \begin{cases} 1 & if\ z_l > \hat{z}_l\ \text{ for some } l\ \in N\setminus\{i,j\} \\ 1 & if\ \ (z_{-ijm} = \hat{z}_{-ijm}) \text{ and } (z_m > \hat{z}_m) \text{ and } [(z_i \geq p+1)\ or\ (z_j \geq p+1)] \\ 0 & \text{otherwise,} \end{cases}$

(where m stands for any direction other than i and j). C_2^m is also ij-symmetric. Notice that in each of the slices defined by $(z_{-ijm} = \hat{z}_{-ijm}\ and\ z_m = \alpha > \hat{z}_m)$, the cost function function C_2^m looks exactly like Figure A.7. Ranking, then, implies the following inequality :

$$\begin{aligned} \varphi_i(C_2^m,q) &= \sum_{z_m=\hat{z}_m+1}^{q_m} f_i(z_m,\hat{z}_{-m}) + \sum_{z_m=\hat{z}_m+1}^{q_m} \sum_{z_j=0}^{p} f_i(p+1,z_j,z_m,\hat{z}_{-ijm}) \\ \leq\ \varphi_j(C_2^m,q) &= \sum_{z_m=\hat{z}_m+1}^{q_m} f_j(z_m,\sigma_{ij}(\hat{z}_{-m})) + \sum_{z_m=\hat{z}_m+1}^{q_m} \sum_{z_j=0}^{p} f_j(z_i,p+1,z_m,\hat{z}_{-ijm}) \end{aligned} \tag{A.16}$$

[3]Notice that all cost increments in the directions i and j occur only in the slice $z_{-ij} = \hat{z}_{-ij}$.

[4]Since all points in the sum satisfy the property that the sum of all coordinates is less than k.

Notice that (A.15) implies that

$$\left\{\begin{array}{l}\sum_{z_m=0}^{q_m} f_i(z_m,\hat{z}_{-m}) = \sum_{z_m=0}^{q_m} f_j(z_m,\sigma_{ij}(\hat{z}_{-m})) \\ \sum_{z_m=0}^{q_m}\sum_{z_j=0}^{p} f_i(p+1,z_j,z_m,\hat{z}_{-ijm}) = \sum_{z_m=0}^{q_m}\sum_{z_j=0}^{p} f_j(z_i,p+1,z_m,\hat{z}_{-ijm})\end{array}\right. \tag{A.17}$$

Furthermore, by induction hypothesis *H2*, one can write

$$\left\{\begin{array}{l}\sum_{z_m=0}^{\hat{z}_m-1} f_i(z_m,\hat{z}_{-m}) = \sum_{z_m=0}^{\hat{z}_m-1} f_j(z_m,\sigma_{ij}(\hat{z}_{-m})) \\ \sum_{z_m=0}^{\hat{z}_m}\sum_{z_j=0}^{p} f_i(p+1,z_j,z_m,\hat{z}_{-ijm}) = \sum_{z_m=0}^{\hat{z}_m}\sum_{z_j=0}^{p} f_j(z_i,p+1,z_m,\hat{z}_{-ijm})\end{array}\right. \tag{A.18}$$

Subtracting (A.18) from (A.17) yields the equalities

$$\left\{\begin{array}{l}\sum_{z_m=\hat{z}_m}^{q_m} f_i(z_m,\hat{z}_{-m}) = \sum_{z_m=\hat{z}_m}^{q_m} f_j(z_m,\sigma_{ij}(\hat{z}_{-m})) \\ \sum_{z_m=\hat{z}_m+1}^{q_m}\sum_{z_j=0}^{p} f_i(p+1,z_j,z_m,\hat{z}_{-ijm}) = \sum_{z_m=\hat{z}_m+1}^{q_m}\sum_{z_j=0}^{p} f_j(z_i,p+1,z_m,\hat{z}_{-ijm})\end{array}\right.$$

which, once substituted into (A.16), give the desired result : $f_i(\hat{z}) \geq f_j(\sigma_{lt}(\hat{z}))$. This ends the first step of induction argument *H2*.

Using the same type of arguments as in the proof of (A.6), we can also show that : $f_i(p+s+1,p-s,\hat{z}_{-ij}) = f_j(p-s,p+s+1,\hat{z}_{-ij}),\ 0 \leq s \leq min\{p,q_1-p-1\}$. This ends the proof of (1.11). ■

We prove (1.12) for $i=1$ and $j=2$, without loss of generality. Fix $(q_1,\overline{z}_2)$ (with $0 \leq \overline{z}_2 \leq q_1$), a direction $t \in N\setminus\{1,2\}$ and consider $\overline{z}_{-12t} \in [0_{n-3},q_{-12t}]$. Just as (A.15), induction hypothesis *H0* implies that :

$$\sum_{z_t=0}^{q_t} f_2(q_1,\overline{z}_2,z_t,\overline{z}_{-12t}) = \sum_{z_t=0}^{q_t}\sum_{z_2=q_1}^{q_2} f_1(\overline{z}_2,z_2,z_t,\overline{z}_{-12t}). \tag{A.19}$$

By induction, suppose that we have :

$$f_2(q_1,\bar{z}_2,z_t,\bar{z}_{-12t}) = \sum_{z_2=q_1}^{q_2} f_1(\bar{z}_2,z_2,z_t,\bar{z}_{-12t}), \forall\, z_t = 0,..,w-1. \tag{A.20}$$

Consider the cost functions :

$$C(z) = \begin{cases} 1 & if\ z_l > \hat{z}_l \text{ for some } l \in N\setminus\{1,2\} \\ 1 & if\ z_{-12t} = \bar{z}_{-12t},\ z_t = w \text{ and } [(z_1,z_2) \geq (q_1,\bar{z}_2) \text{ or } (z_1,z_2) \geq (\bar{z}_2),q_1)] \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bar{C}(z) = \begin{cases} 1 & if\ z_l > \hat{z}_l \text{ for some } l \in N\setminus\{1,2\} \\ 1 & if\ z_{-ijt} = \bar{z}_{-12t},\ z_t \geq w+1 \text{ and } [(z_1,z_2) \geq (q_1,\bar{z}_2) \text{ or } (z_1,z_2) \geq (\bar{z}_2,q_1)] \\ 0 & \text{otherwise.} \end{cases}$$

Just as we did in the two previous pages (with C_1 and $C_2^{m'}$), applying Ranking to C and $\bar{C}$ and combining (A.19) and (A.20) yields the desired result :

$$\sum_{z_2=q_1}^{q_2} f_1(\bar{z}_2,z_2,z_t = w,\bar{z}_{-12t}) = f_2(q_1,\bar{z}_2,z_t = w,\bar{z}_{-12t}). \qquad \blacksquare$$

FIG. A.2 – The cost functions used in the proof

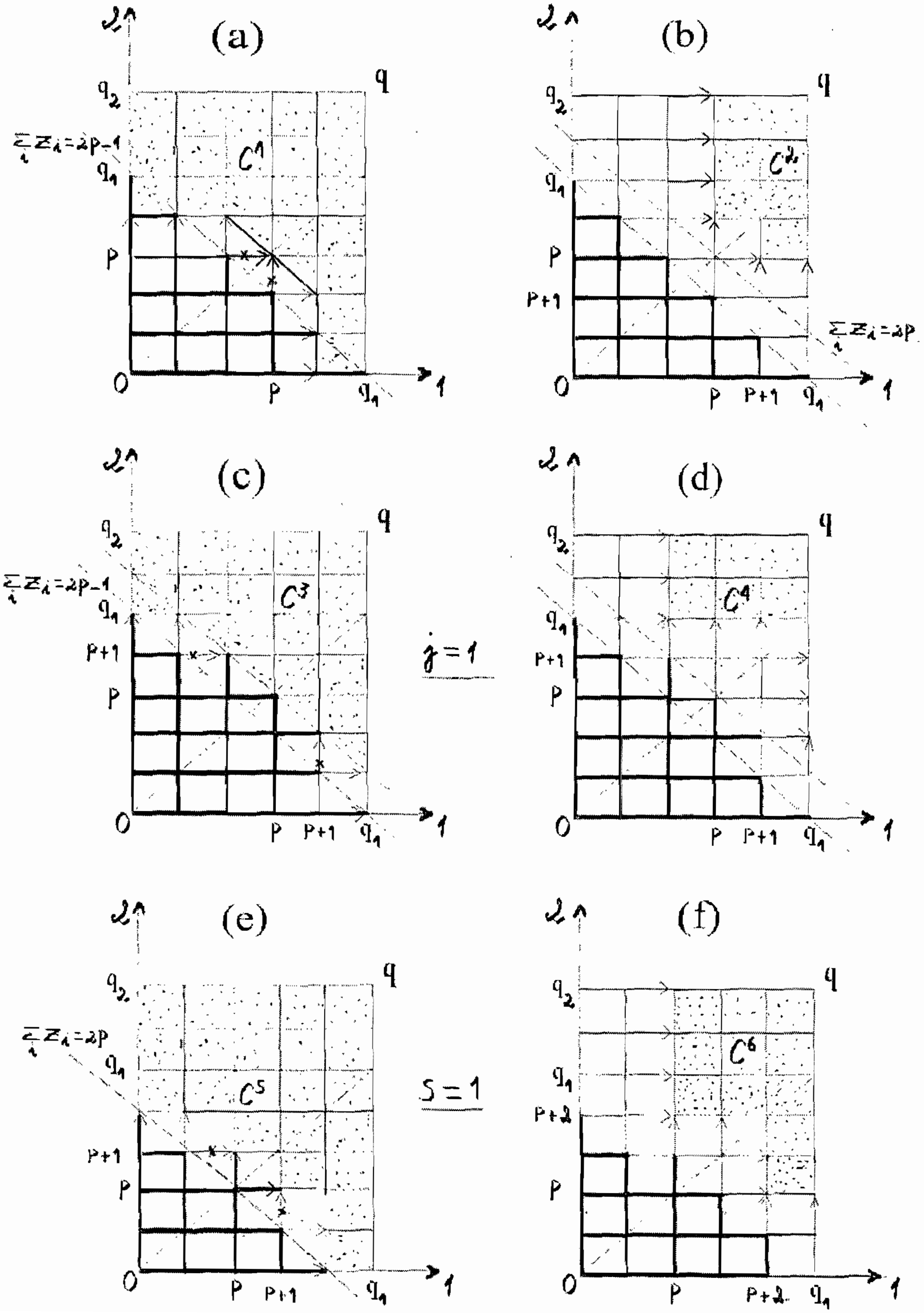


FIG. A.3 – $\underline{Fr}(X)$ and $\overline{Fr}(X)$; f_+ and f_-

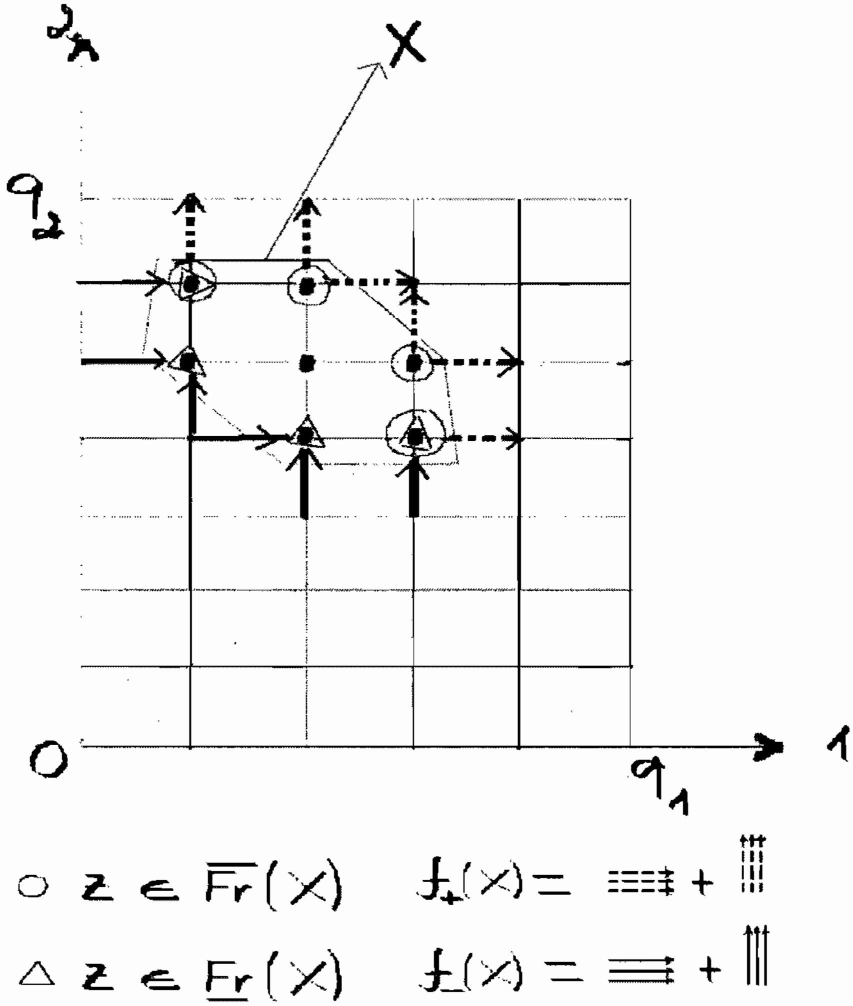


FIG. A.4 – The three possible cases

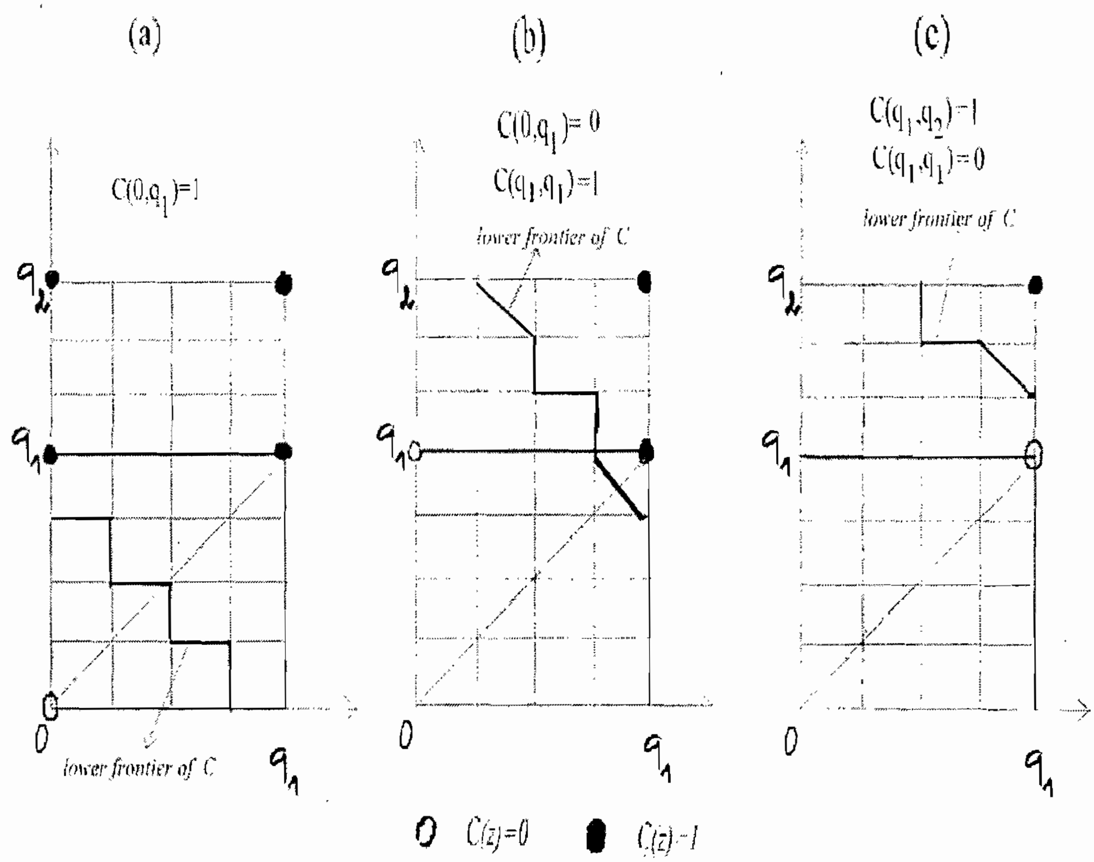
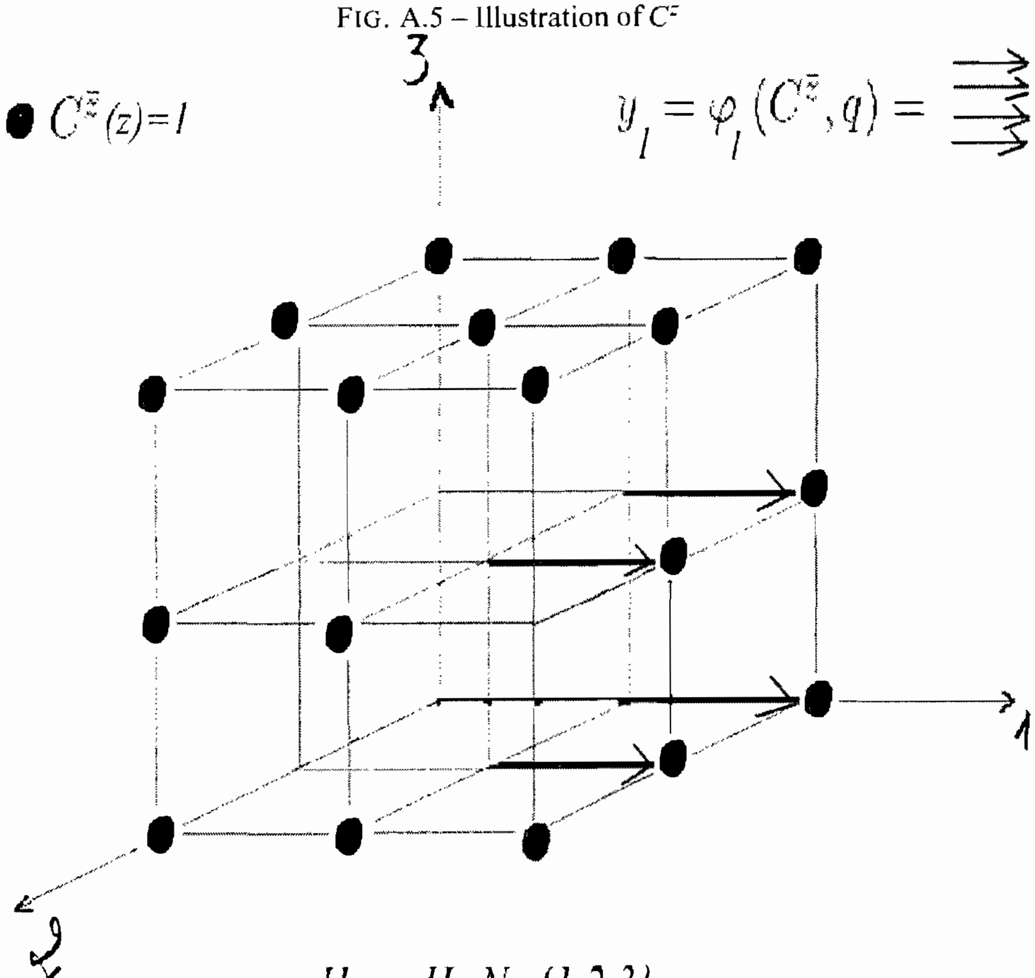


FIG. A.5 – Illustration of $C^{\bar{z}}$



Here $H=N=\{1,2,3\}$,
$q=(2,2,2)$ and $\bar{z}=2$.

FIG. A.6 – Illustration of $C'^{\bar{z}}$

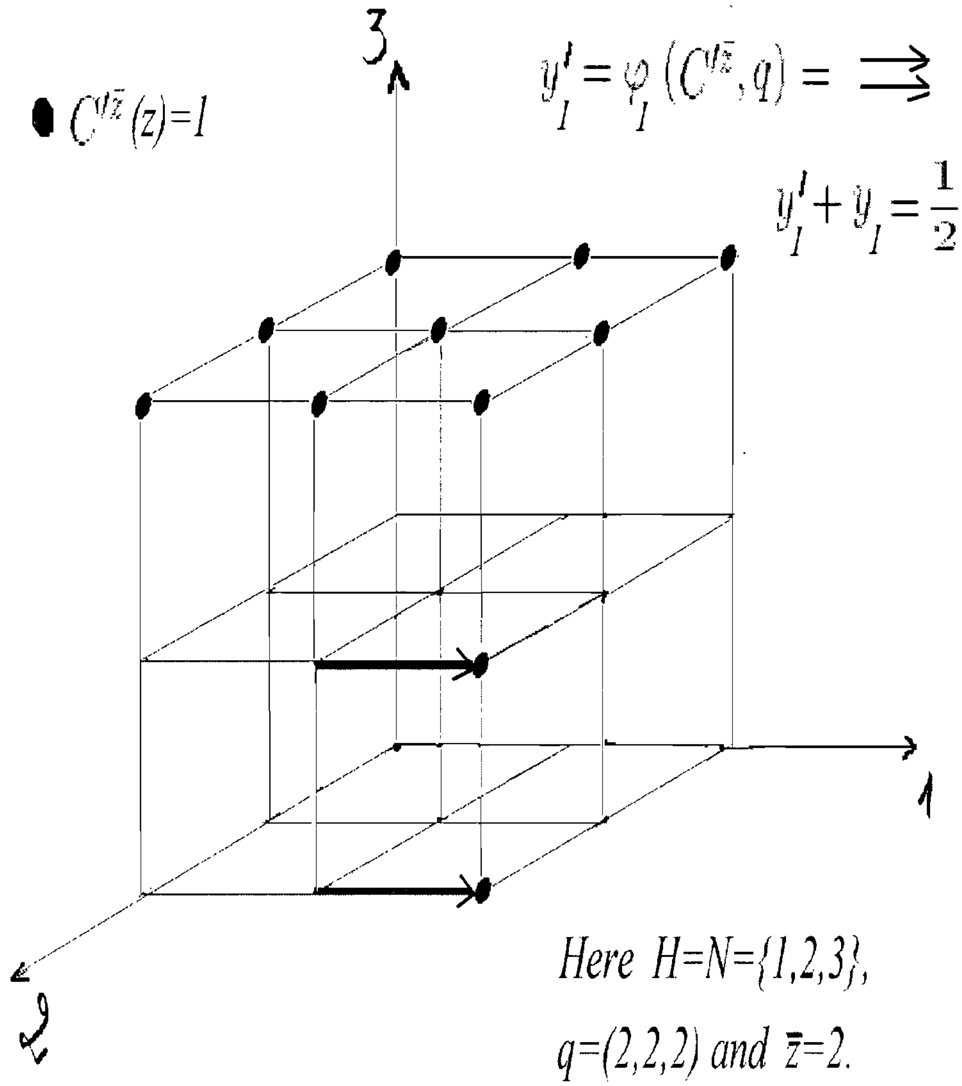
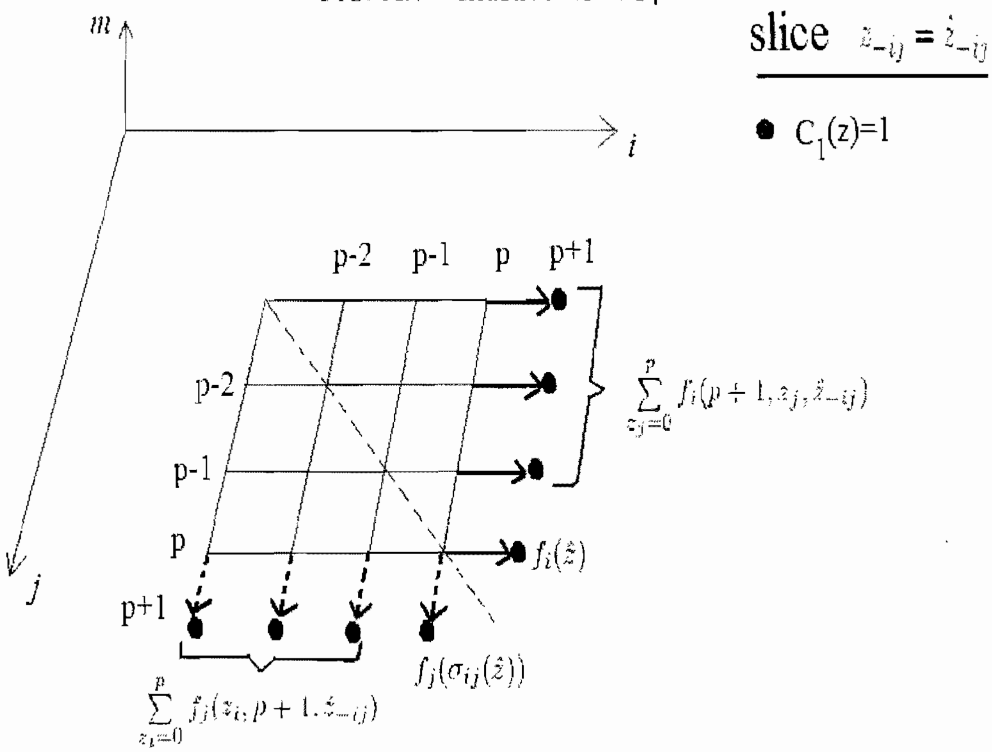


FIG. A.7 – Illustration of C_1



$\sum_{z_j=0}^{p} f_i(p+1, z_j, \hat{z}_{-ij}) = \sum_{z_i=0}^{p} f_j(z_i, p+1, \hat{z}_{-ij})$ by induction hypothesis *H1*.

ANNEXE B

APPENDIX TO CHAPTER 2

B.1 Monotonicity and concavity of the utility $V_0(X)$ of the reserves

In this part, we consider the value function associated to the problem described by (2.8). $V_0(X)$ represents the welfare enjoyed by the importing country if it has accumulated X units of oil prior to the disruption. We want to prove that $V_0(X)$ is increasing and concave.

We show first that (*at each date t*) the optimal consumption $q^*(t)$ increases with the accumulated stock X. Indeed, by condition (2.17), we know that :

$$\int_0^{+\infty} u'^{-1}(\bar{\lambda} e^{(r+\theta_b)t})\, dt = X\,. \tag{B.1}$$

Since u'^{-1} is decreasing (because u' is), it follows that $\frac{d\bar{\lambda}}{dX} < 0$. Next, taking the derivative with respect to X, (2.16) yields :

$$\frac{dq^*(t)}{dX} = e^{(r+\theta_b)t} \underbrace{\frac{1}{u''(q^*(t))}}_{<0} \underbrace{\frac{d\bar{\lambda}}{dX}}_{<0} > 0\,. \tag{B.2}$$

It follows that V_0 also increases with the reserves X. Indeed, from what precedes, we have :

$$V_0'(X) = \int_0^{+\infty} e^{-(r+\theta_b)t} \underbrace{\frac{dq^*(t)}{dX}}_{>0} \underbrace{u'(q^*(t))}_{>0} dt > 0\,. \tag{B.3}$$

Now, consider $X_1 < X_2$ and denote by $\{q_1^*(t), t > 0\}$ and $\{q_2^*(t),\ t > 0\}$ the optimal consumption paths corresponding to these stocks. By what we have just shown, we can write : $q_1^*(t) < q_2^*(t)$, for all $t > 0$. To prove the concavity of $V_0(X)$, it is sufficient to show that for an infinitesimal increment dX of the reserves, we have :

$$V_0(X_1+dX)-V_0(X_1) > V_0(X_2+dX)-V_0(X_2)\ . \tag{B.4}$$

Let $\{q_2^*(t)+\xi(t), t>0\}$ be the optimal consumption path associated to X_2+dX. Notice that $\xi(t)>0$ at each $t>0$, since optimal consumption q^* increases with the stock X. Applying equation (2.15), respectively to X_2+dX and X_2, and subtracting the latter from the former gives :

$$\int_0^{+\infty}\xi(t)dt = dX\ . \tag{B.5}$$

Since u is concave and $q_1^*(t)<q_2^*(t)$, we have :[1]

$$u(q_1^*(t)+\xi(t))-u(q_1^*(t)) > u(q_2^*(t)+\xi(t))-u(q_2^*(t)) \quad \text{for all } t>0\ .$$

It follows that :

$$\int_0^{+\infty} e^{-(r+\theta_b)t}\,[u(q_1^*(t)+\xi(t))-u(q_1^*(t))]dt > \int_0^{+\infty} e^{-(r+\theta_b)t}\,[u(q_2^*(t)+\xi(t))-u(q_2^*(t))]dt\ .$$

Therefore, we can write :

$$\int_0^{+\infty} e^{-(r+\theta_b)t}\,u(q_1^*(t)+\xi(t))dt - V_0(X_1) > V_0(X_2+dX)-V_0(X_2)\ . \tag{B.6}$$

Finally, from the equality

$$\int_0^{+\infty}(q_1^*(t)+\xi(t))dt = \underbrace{\int_0^{+\infty} q_1^*(t)dt}_{=X_1} + \underbrace{\int_0^{+\infty}\xi(t)dt}_{=dX} = X_1+dX,$$

we can claim that $\{q_1^*(t)+\xi(t))\}$ is a feasible consumption path, given the stock X_1+dX

[1]Notice that $\xi(t)$ are also infinitesimal increments. They are even smaller than dX, since they are all positive and they sum up to dX (see equation (B.5)).

at the beginning of the embargo. Hence, one can write :

$$V_0(X_1+dX) \geq \int_0^{+\infty} e^{-(r+\theta_b)t}\, u(q_1^*(t)+\xi(t))dt\,.$$

This, substituted into (B.6), gives the desired result.

B.2 The example of a constant EIS utility function

Suppose that $u(q(t)) = \frac{q^{1-\sigma}}{1-\sigma}$. From (2.17), one gets :

$$\int_0^{+\infty} \left[e^{(r+\theta_b)t}\,\bar{\lambda}\right]^{-1/\sigma} dt = X_0\,. \tag{B.7}$$

Therefore, the value of the shadow price is :

$$\bar{\lambda} = \left[\frac{r+\theta_b}{\sigma}X_0\right]^{-\sigma}. \tag{B.8}$$

Finally, by equation (2.16), the optimal SPR depletion after the embargo has happened is given by :

$$q^*(t) = \frac{r+\theta_b}{\sigma}X_0e^{-\frac{r+\theta_b}{\sigma}t}\,. \tag{B.9}$$

During the embargo, the stock at each date t is then given by $X^*(t) = X_0e^{-\frac{r+\theta_b}{\sigma}t}$, which is a decreasing function of both r and θ_b. This proves proposition 2.3.1. Also, since the absence of the backstop corresponds to the case where $\theta_b \to 0$, taking the limit yields :

$$\lim_{\theta_b \to 0} X^*(t) = X_0e^{-\frac{r}{\sigma}t} > X_0e^{-\frac{r+\theta_b}{\sigma}t}, \text{ for all } \theta_b > 0 \text{ and all } t > 0\,.$$

Thus, if country B is not expecting a backstop, it will hold higher precautionary reserves at any time.

The value function (see equation (2.9)) obtains by computing the welfare associated to this path. Performing this calculation gives :

$$V_0(X_0) = \left(\frac{\sigma}{r+\theta_b}\right)^{\sigma} \frac{X^{1-\sigma}}{1-\sigma} + \frac{Z}{r+\theta_b} + \frac{\theta_b U^*}{r(r+\theta_b)}, \tag{B.10}$$

where $U^* = u(u'^{-1}(c)) + Z - cu'^{-1}(c)$.

Since the evolution of the price is given by $p(t) = p(0)e^{rt}$, using the no-arbitrage condition given by (2.27) and replacing π *and* $\bar{\pi}$ by their values (2.19) yields the *desired level of the stock* at each time t.

$$X^d(t) = \frac{\sigma}{(r+\theta_b)\{p(t)(1+\frac{\theta_b^2}{\theta_0^2}e^{(\theta_0-\theta_b)t})\}^{1/\sigma}} \tag{B.11}$$

$X^d(t)$ represents the amount of SPR that the importing country would acquire if its budget Z was sufficiently large to purchase the stock $X^d(0)$ from date 0.

Performing some comparative statics shows that $\frac{\partial X(t)^*}{\partial r}$, $\frac{\partial X^*(t)}{\partial t}$, $\frac{\partial X^*(t)}{\partial \theta_b} < 0$; confirming the mitigating effect of the backstop on the embargo threat.

Also, equation (B.11) shows that : $\frac{\partial X^*(t)}{\partial \theta_0} > 0 \Leftrightarrow \frac{\partial(e^{\theta_0 t}/\theta_0^2)}{\partial \theta_0} = \frac{1}{\theta_0^2}(t - \frac{1}{\theta_0})e^{\theta_0 t} < 0$.

This proves Proposition 2.3.3 : a slight decrease in the expected date of the embargo entails higher (resp. lower) reserves at any date t such that $t < E(\tau_0) = \frac{1}{\theta_0}$ (resp. $t > E(\tau_0) = \frac{1}{\theta_0}$).

B.3 Proof of $\frac{\partial V_s}{\partial y_s}(y_s = 0) > 0$

Recall that $\theta_b \equiv y$. First, we prove that $\frac{\partial V_0}{\partial y_0}(y_0 = 0) > 0$. This means that, regardless of the cost, country B benefits from choosing a positive effort during the embargo.

By the envelope theorem, we have :[2]

$$\frac{\partial V_0(y_0, X)}{\partial y_0} = \int_0^{+\infty} -te^{-(r+y_0)t}\,[U(q^*(t), Z)]\,dt + \int_0^{+\infty} e^{-y_0 t}(1 - y_0 t)V_b(t)\,dt\,. \tag{B.12}$$

[2] $V_b(t)$ is expressed in discounted value.

Rewriting (B.12) in the particular case where $y_0 = 0$ yields :

$$\frac{\partial V_0(y_0=0,X)}{\partial y_0} = \int_0^{+\infty} -te^{-rt}\, U(q^*(t),Z)\, dt + \int_0^{+\infty} V_b(t)\, dt \ .$$

Integrating by parts :

$$\frac{\partial V_0(y_0=0,X)}{\partial y_0} = \underbrace{[-te^{-rt}U(q^*(t),Z)]_0^{+\infty}}_{=0} - \int_0^{+\infty}(\int_t^{+\infty} e^{-rs}\, U(q^*(s),Z)ds)dt + \int_0^{+\infty} V_b(t)\, dt$$

$$= \int_0^{+\infty}[V_b(t) - \int_t^{+\infty} e^{-rs}\, U(q^*(s),Z)ds]\, dt.$$

Notice that, on the optimal path, from the moment t at which the backstop occurs, country B has the possibility to use the remaining stock $X^*(t)$ instead of the backstop, in which case (discounted) welfare would be given by $\int_t^{+\infty} e^{-rs}\, U(q^*(s),Z)ds$. The fact that the country picks the backstop entails that we necessarily have : $V_b(t) > \int_t^{+\infty} e^{-rs}\, U(q^*(s),Z)ds$, which proves that $\frac{\partial V_0}{\partial y_0}(y_0 = 0, X) > 0$.
In addition, by equation (B.12), we can write $\lim\limits_{y_0 \to +\infty} \frac{\partial V_0(y_0,X)}{\partial y_0} = 0$.

Using the same procedure, we can also prove that $\frac{\partial V_1}{\partial y_1}(y_1 = 0) > 0$ and $\lim\limits_{y_1 \to +\infty} \frac{\partial V_1(y_1,X)}{\partial y_1} = 0$.

B.4 Proof of $\frac{\partial^2 V_s}{\partial y_s \partial X(\tau_s)} < 0$

Replacing θ_b by y_0 in (B.3) gives :

$$\frac{\partial V_0}{\partial X(\tau_0)} = \int_0^{+\infty} e^{-(r+y_0)t}\, \underbrace{\frac{dq^*(t)}{dX}}_{>0}\, \underbrace{u'(q^*(t))}_{>0} dt > 0.$$

By the envelope theorem, we have :

$$\frac{\partial^2 V_0}{\partial y_0 \partial X_0} = \frac{\partial^2 V_0}{\partial X_0 \partial y_0} = \int_0^{+\infty} -te^{-(r+y_0)t}\, \underbrace{\frac{dq^*(t)}{dX}}_{>0}\, \underbrace{u'(q^*(t))}_{>0} dt < 0\ .$$

By a similar argument, we can also prove that $\frac{\partial^2 V_1}{\partial y_1 \partial X(\tau_1)} < 0$. Notice that we assume the regularity conditions needed to ensure that the cross derivatives coincide.

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